Distributed Power Allocation in Multi-User Multi-Channel Cellular Relay Networks

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***Abstract*—In this paper, we consider the amplify-and-forward relaying transmission in the downlink of a multi-channel cellular network with one base station and multiple relay-destination pairs. Spatial reuse of the relaying slot by allowing simultaneous transmissions from the relays is adopted to avoid the spectral loss incurred by the half-duplex relays. The relays are modeled as rational agents engaging in a *non-cooperative* game. In order to maximize its individual rate, each relay node iteratively allocates its power across different subchannels based on local information, while treating the signals from the other users as additive noise. First, we propose a distributed algorithm based on best response that is applicable in any signal to interference plus noise ratio (SINR) regions. Then, by focusing on the low SINR region, we propose a modified iterative water-filling algorithm. The existence of Nash equilibrium (NE) is guaranteed and the sufficient condition to reach a NE iteratively is determined. Next, we consider medium to high SINR regions and propose a distributed algorithm based on the sub-optimal response, which can be shown to reduce to the classic Gaussian interference channel model, for which analytical sufficient conditions for the convergence to the unique NE can be readily obtained. Finally, we extend the analysis to a general network topology wherein the users having different channel conditions coexist. The results show that, in low SINR regions, the proposed modified iterative water-filling algorithm yields a higher average sum rate than two simplified algorithms, i.e., the equal power allocation scheme and the conventional time-division based protocol, while in medium to high SINR regions, the sub-optimal-response based algorithm outperforms these two simplified algorithms in terms of the average sum rate.**

***Index Terms*—Distributed power allocation, interference channel, relay networks, game theory, Nash equilibrium.**

I. Introduction

In many wireless networks, the transmitters may not be able to support multiple physical antennas due to limitations in size, complexity, cost or other constraints. Cooperative communication [1] is an alternative approach that is becoming increasingly popular thanks to its ability to provide spatial diversity without packing multiple antennas physically into small-size mobile nodes. Depending on whether the source's signals are forwarded by the relay after regeneration or not, user cooperation can be further classified as non-regenerative cooperation, e.g., amplify-and-forward (AF), and regenerative cooperation, e.g., decode-and-forward (DF) [1].

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It has been shown that power allocation is an efficient approach to enhance the spectral efficiency. Several power allocation algorithms were proposed, by utilizing the channel state information (CSI) at the transmitter, to improve the performance of cooperative networks. In particular, the authors in [2] proposed *opportunistic cooperation* which can achieve the minimum outage probability for cooperative DF networks by dynamically adjusting the transmission time and corresponding power of each network node based on the global CSI. In [12], the authors investigated the rate optimization problem in multiple access AF relay networks based on a centralized power control scheme. Unfortunately, these centralized algorithms are, in general, difficult to implement in practice despite their appealing performance gain, due to the high signaling overhead incurred in coordination, information exchange, etc. [16]. Another drawback of the conventional relay transmission is the spectral loss due to the pre-log factor of $1/2$ [1]. Consequently, spectrally-efficient transmission protocols have been proposed to avoid the loss [3][4]. For instance, the authors in [3] proposed a two-way relaying protocol that allows the source nodes to transmit simultaneously to enhance the utilized spectral efficiency. A sophisticated protocol was proposed in [4], wherein the source node first transmits in non-overlapping time slots to different relays that will transmit simultaneously in the remaining time slot. By doing so, the relaying time slot is spatially reused and thus, the spectral loss factor is avoided. As in [2], however, the proposed power allocation algorithms require the global CSI in most existing works on spectrally-efficient relay networks, e.g., [3]. Furthermore, in these papers, the users need to explicitly cooperate with each other, which may not always be realistic when the participating users are self-interested.

To address the aforementioned concerns, some distributed power allocation algorithms were recently proposed (see [5][8]-[10] and references therein). In [8], the authors considered an interference-free wireless AF network and proposed two power allocation strategies to maximize either the minimum signal-to-noise ratio (SNR) among all the users or the weighted sum of SNRs using both centralized and distributed algorithms. One fundamental assumption in [8] is that all the users are willing to *cooperatively* maximize the network utility, which implicitly requires coordination among the users. In a network with self-interested users that care only about their own utilities, it is of particular interest to model the problem of distributed power allocation from the game-theoretic perspective [16]. For example, the problem of allocating the relay's power to different sources based on auctions was examined in [9] for a wireless AF relay network. However, as in conventional relay networks, the spectral loss issue was not addressed in [9]. In [10], an iterative power allocation algorithm was proposed to optimize each individual user's rate for a simple DF network with only two selfish users. Considering a spectrally-efficient network with spatial reuse of the relaying slot, the authors focused on the energy efficiency of the network and investigated the distributed power allocation problem using game theory [5]. Nonetheless, neither the rate-maximization problem nor the convergence of the proposed algorithm was discussed in [5].

In this paper, we consider a spectrally-efficient cellular relay network wherein concurrent transmissions from different relays are allowed, and derive distributed power allocation algorithms for self-interested relay nodes using the framework of non-cooperative game theory [30]. Each relay node iteratively maximizes its individual rate based on the local information by allocating its power across different subchannels, while treating the signals from the other users as additive noise. First, we propose a distributed algorithm based on the best response, which is applicable in any signal to interference plus noise ratio (SINR) regions but lacks mathematical tractability. To analyze the convergence of the algorithm, we then consider low SINR regions and propose a simultaneous modified iterative water-filling algorithm. Specifically, each user updates its power allocation by multiplying the water level by a certain factor on a per-subchannel basis. The existence of Nash equilibrium (NE) is guaranteed and the sufficient condition to reach a NE is also determined. Next, by focusing on medium to high SINR regions, we propose a distributed algorithm based on the sub-optimal response. With a negligible performance loss in terms of the average sum rate of all the users compared to the best-response based algorithm, the algorithm based on the sub-optimal response is mathematically tractable and easier to compute. Furthermore, it can be equivalently viewed as the classic Gaussian interference channel model, for which analytical sufficient conditions for the convergence to the unique NE can be readily obtained. Finally, we extend the analysis to a general network topology, wherein the users having different channel conditions coexist, and conduct extensive simulations to validate the analysis.

The rest of this paper is organized as follows. Section II describes the system model and problem formulation. In Section III, distributed power allocation algorithms are proposed for the multi-user relay network. To analyze the convergence of the proposed algorithms, we first separately consider low SINR and medium to high SINR regions, and then extend the analysis to general network topologies. Simulation results are shown in Section IV. Finally, concluding remarks are offered in Section V.

II. System Model and Problem Formulation

Consider a cellular relay network consisting of one base station and Q relay-destination pairs[1], as illustrated in Fig. 1 [4]-[7].

A. System Model

The i-th relay and destination nodes are indexed by $\mathcal{R}_i$ and $\mathcal{D}_i$, respectively, for $i = 1, 2 \cdots Q$, and the base station

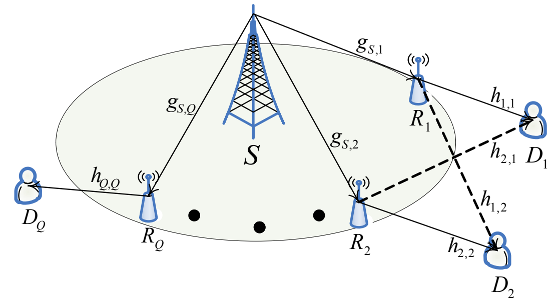


Fig. 1. Network model.

is represented by $\mathcal{S}$. Assuming that the system has N unit-bandwidth subchannels (or frequency bins), we denote the channel coefficients for the k-th $\mathcal{S} - \mathcal{R}_j$ and the k-th $\mathcal{R}_i - \mathcal{D}_j$ subchannels by $g^k_{\mathcal{S},j}$ and $h^k_{i,j}$, respectively, for $k = 1, 2 \cdots N$ and $i, j = 1, 2 \cdots Q$. The transmit powers of $\mathcal{S}$ and $\mathcal{R}_i$ over the k-th subchannel are $P_{\mathcal{S},k}$ and $P_{i,k}$, respectively. Like in [8], we assume that the power of the base station, $P_{\mathcal{S},k}$, is predetermined, and concentrate only on the adaptation of the relays' power in this paper. The transmission between different nodes suffers from frequency nonselective independent ergodic block (or quasi-static) fading. Local CSI, i.e., $g^k_{\mathcal{S},i}$ and $h^k_{i,i}$, is only available at user i through training sequences and channel feedback [2]. Hence, due to the distributed nature of the considered communication problem, neither $g^k_{\mathcal{S},j}$, $h^k_{i,j}$ nor $h^k_{j,l}$ is known to $\mathcal{R}_i$ or $\mathcal{D}_i$ if $j \neq i$, for $l = 1, 2 \cdots N$. Furthermore, we assume the zero-mean complex additive white Gaussian noise (AWGN) at each node over each subchannel to have a variance[2] of N_0. Due to the half-duplex constraint, we consider orthogonal relaying transmissions, e.g., the source node and the relay node transmit in two non-overlapping time slots. For the convenience of analysis, the direct link between S and D_i is neglected due to, for instance, the shadowing effects [3]. To forward the data from the base station to the destination, we adopt the classical AF strategy [1] as the relaying operation, which has been shown to be an appealing technique due to its low cost and easy implementation as opposed to DF [11].

Considering the downlink transmission in a time-division-multiple-access (TDMA) cellular relay network, we aim at improving the spectral efficiency by spatially reusing the relaying slot. To this end, we divide each frame equally into $Q+1$ orthogonal time slots. During the i-th slot of the frame, the base station transmits to $\mathcal{R}_i$, for $i = 1, 2 \cdots Q$. Then, simultaneous transmissions from all the relays are activated during the $(Q+1)$-th slot of the frame. Hence, a spectral efficiency factor of $1/(Q+1)$ can be effectively achieved [5], unlike in the conventional TDMA-based relay network wherein the relays do not transmit simultaneously and the achievable rate is scaled by $1/2Q$ due to the spectral loss. Moreover, the signaling overhead in coordinating the relays' transmission in the conventional TDMA protocol is no longer

[1]Throughout this paper, we interchangeably use the term "user i" to represent the i-th relay-destination pair $\mathcal{R}_i - \mathcal{D}_i$.

[2]The analysis throughout this paper can also be easily generalized to the case that the AWGN power at the k-th subchannel is N_0^k, whereas $N_0^i \neq N_0^j$ if $i \neq j$. For the convenience of notation, we assume that $N_0^i = N_0^j = N_0$ for $i, j = 1, 2 \cdots N$, as in [17].

required. A similar transmission scheme for a multiple-antenna system was considered in [5], wherein the authors focused on the energy efficiency of individual relays. For the network considered here, the signals received at $\mathcal{R}_i$ and $\mathcal{D}_i$ over the k-th subchannel can then be written, respectively, as

$$y^k_{\mathcal{R}_i} = g^k_{\mathcal{S},i}\sqrt{P_{\mathcal{S},k}}x_{i,k} + n^k_{\mathcal{R}_i} \text{ and } y^k_{\mathcal{D}_i} = \sum_{j=1}^{Q} h^k_{j,i}\alpha^k_{\mathcal{R}_j} y^k_{\mathcal{R}_j} + n^k_{\mathcal{D}_i}, \quad (1)$$

where $x_{i,k}$ is the unit-variance transmit signal from S to D_i, $\alpha^k_{\mathcal{R}_i}$ is the *amplification factor* of $\mathcal{R}_i$, $n^k_{\mathcal{R}_i}$ and $n^k_{\mathcal{D}_i}$ are the statistically-independent AWGN terms at $\mathcal{R}_i$ and $\mathcal{D}_i$ over the k-th subchannel, respectively. The amplification factor $\alpha^k_{\mathcal{R}_i}$, which is part of the local information known by user i, is chosen to satisfy the power constraint at the relay, i.e., $\alpha^k_{\mathcal{R}_i} = \sqrt{\frac{P_{i,k}}{|g^k_{\mathcal{S},i}|^2 P_{\mathcal{S},k} + N_0}}$. By plugging $y^k_{\mathcal{R}_i} = g^k_{\mathcal{S},i}\sqrt{P_{\mathcal{S},k}}x_{i,k} + n^k_{\mathcal{R}_i}$ into (1), we can rewrite the received signal over the k-th subchannel at $\mathcal{D}_i$ as

$$\begin{aligned} y^k_{\mathcal{D}_i} = {} & h^k_{i,i}\alpha^k_{\mathcal{R}_i} g^k_{\mathcal{S},i}\sqrt{P_{\mathcal{S},k}}x_{i,k} + \sum_{j\neq i}^{Q} h^k_{j,i}\alpha^k_{\mathcal{R}_j} g^k_{\mathcal{S},j}\sqrt{P_{\mathcal{S},k}}x_{j,k} \\ & + \sum_{j=1}^{Q} h^k_{j,i}\alpha^k_{\mathcal{R}_j} n^k_{\mathcal{R}_j} + n^k_{\mathcal{D}_i}. \end{aligned} \quad (2)$$

Since we focus on developing distributed algorithms, we assume that $\mathcal{D}_i$ is only interested in the signal forwarded by $\mathcal{R}_i$ and consider transmission techniques with no interference cancelation. Treating $h^k_{i,i}\alpha^k_{\mathcal{R}_i} g^k_{\mathcal{S},i}\sqrt{P_{\mathcal{S},k}}x_{i,k}$ as the desired signal component and the multiuser interference as noise at $\mathcal{D}_i$ and after some mathematical manipulations, we can then express the receive SINR at $\mathcal{D}_i$ over the k-th subchannel as

$$\gamma_{\mathcal{D}_i,k} = \frac{|g^k_{\mathcal{S},i}|^2 |h^k_{i,i}|^2 P_{\mathcal{S},k} P_{i,k}}{|h^k_{i,i}|^2 N_0 P_{i,k} + \left(|g^k_{\mathcal{S},i}|^2 P_{\mathcal{S},k} + N_0\right)\cdot \Delta_{i,k}}, \quad (3)$$

where $\Delta_{i,k} = \sum_{j=1,j\neq i}^{Q} |h^k_{j,i}|^2 P_{j,k} + N_0$. Assuming that the base station transmits information streams across all the subchannels, the achievable rate of user i, for $i = 1, 2 \cdots Q$, can be mathematically expressed as[3]

$$R_i(\mathbf{p}_i;\mathbf{p}_{-i}) = \frac{1}{Q+1}\sum_{k=1}^{N}\log\left(1+\gamma_{\mathcal{D}_i,k}\right) \quad (4)$$

where the scaling factor $1/(Q+1)$ is due to the fact that $\mathcal{S}$ transmits $x_{i,k}$ only for a fraction of $1/(Q+1)$ of the frame, $\gamma_{\mathcal{D}_i,k}$ is defined in (3), $\mathbf{p}_i = [P_{i,1}, P_{i,2}\cdots P_{i,N}]^{\mathrm{T}}$, and $\mathbf{p}_{-i} = (\mathbf{p}_1\cdots\mathbf{p}_{i-1}, \mathbf{p}_{i+1}\cdots\mathbf{p}_Q)$.

Before proceeding to the problem formulation, we comment on how this paper significantly differs from the existing works on Gaussian interference channels, despite the absence of direct channels. The key difference is that the signal forwarded by the relay node is not "clean", whereas the source transmits noiseless signals to the destination in single-hop Gaussian interference channels, i.e., the relay amplifies the Gaussian noise, in addition to the desired signal, which can be seen from the signal model in (2). Due to the noise propagation by the relay, the term $P_{i,k}$, i.e., the i-th relay's power over the k-th sub-channel, appears both in the numerator and denominator of the receive SINR. Nonetheless, the denominator in the receive SINR in Gaussian interference channels only contains the multi-user interference and Gaussian noise, which significantly simplifies the expression of the receive SINR and analysis of distributed power allocation algorithms. Hence, the analysis in this paper can be regarded as a generalization of the existing results on Gaussian interference channels. As a special case, if the signal forwarded by the relay is noiseless or "clean" (i.e., the base station to relay channel is almost "perfect" or equivalently $|g_{i,k}|^2 \to \infty$), the dual-hop relay channel reduces to the conventional Gaussian interference channel and the receive SINR becomes $\frac{|h^k_{i,i}|^2 P_{i,k}}{\sum_{j=1,j\neq i}^{Q}|h^k_{j,i}|^2 P_{j,k}+N_0}$, which can also be obtained by taking the limit of (3) with respect to $|g_{i,k}|^2 \to \infty$.

B. Problem Formulation

The traditional view of interference channels focuses on interference cancelation by allowing the transmitters to cooperatively transmit their signals to the destination. Unfortunately, to achieve this, a central controller having the global CSI knowledge is required, thereby incurring a heavy spectral loss due to the signalling overhead required for the information exchange and coordination messages. If such global cooperation, except for time synchronization, cannot be assumed, each user may try to compete for the scarce resource and optimize its own performance based on its local information, regardless of all the other users. In this communication setting, a natural solution is to formulate the interference channel into the framework of *non-cooperative* game theory [16]. In particular, we consider a non-cooperative game in which each relay node optimally allocates its power across all the subchannels to maximize its own payoff, i.e., achievable rate in (4). The power-allocation game can be mathematically stated in the following structure

$$\mathcal{G} = \{\Omega, \{\mathcal{P}_i\}_{i\in\Omega}, \{R_i(\mathbf{p}_i;\mathbf{p}_{-i})\}_{i\in\Omega}\}, \quad (5)$$

where $\Omega \triangleq \{1, 2\cdots Q\}$ is the set of active users (i.e., $\mathcal{R}_i - \mathcal{D}_i$ pair), $\mathcal{P}_i$ is the set of admissible power allocation strategies of user i defined as

$$\mathcal{P}_i = \left\{\mathbf{p}_i \in \mathbb{R}^N : \sum_{k=1}^{N} P_{i,k} = P_i^{\max}, \; 0 \le P_{i,k} \le P_{i,k}^{\max}, \forall k \in \{1, 2\cdots N\}\right\}, \quad (6)$$

and $R_i(\mathbf{p}_i;\mathbf{p}_{-i})$ is the payoff function of user i, given its own power allocation $\mathbf{p}_i$ and that of all the other users $\mathbf{p}_{-i}$. Note that $\sum_{k=1}^{N} P_{i,k}^{\max} \ge P_i^{\max}$ is assumed for all $i \in \Omega$ to ensure that $\mathcal{P}_i$ is always nonempty. Moreover, the equality $\sum_{k=1}^{N} P_{i,k} = P_i^{\max}$ in the admissible strategies can also be replaced by $\sum_{k=1}^{N} P_{i,k} \le P_i^{\max}$ without affecting the optimal solution, since the equality must be activated at the optimum point, which can be easily proved using standard optimization

[3]All logarithms have a base of two throughout this paper, unless otherwise stated.

techniques [25]. Given the power level of all the other users, the optimal power allocation strategy of user i is referred to as the *best response* function denoted by $\mathbf{p}_i^*$, for all $i \in \Omega$. In the non-cooperative game, the NE is achieved when user i, given the strategy of all the other users $\mathbf{p}_{-i}$, cannot increase its payoff $R_i(\mathbf{p}_i;\mathbf{p}_{-i})$ by unilaterally changing its own power allocation strategy $\mathbf{p}_i$, for all $i \in \Omega$. Mathematically, the NE, denoted by $\mathbf{p}^*$, of the game $\mathcal{G}$ in (5) is formally defined as follows [30]:

$$R_i(\mathbf{p}_i^*;\mathbf{p}_{-i}^*) \geq R_i(\mathbf{p}_i;\mathbf{p}_{-i}^*), \quad \forall\, \mathbf{p}_i \in \mathcal{P}_i,\ \forall\, i \in \Omega\,. \tag{7}$$

III. Distributed Power Allocation Algorithms

In this section, we first develop a best response based iterative power allocation algorithm wherein each user myopically maximizes its achievable rate. Then, due to the mathematical intractability of the best response based algorithm, we resort to commonly used approximation techniques and propose two sub-optimal distributed algorithms which are applicable in low SINR and medium to high SINR regions, respectively. We also show that these two sub-optimal algorithms can be combined to address the general network conditions.

A. Best Response Based Algorithm

Focusing on distributed power allocation algorithms, we first formulate the rate-maximization problem, for each user $i \in \Omega$, within the framework of non-cooperative game theory as

$$\begin{aligned}\max_{\mathbf{p}_i} R_i(\mathbf{p}_i;\mathbf{p}_{-i}) = \max_{\mathbf{p}_i} \sum_{k=1}^{N} \frac{1}{Q+1} \log\left(1+\gamma_{\mathcal{D}_i,k}\right) \\ \text{s.t.} \quad \mathbf{p}_i \in \mathcal{P}_i,\end{aligned} \tag{8}$$

where $\gamma_{\mathcal{D}_i,k}$ given in (3) is the receive SINR over the k-th subchannel at $\mathcal{D}_i$. Based on the fact that the problem in (8) is concave in $\mathbf{p}_i$ given any feasible value of $\mathbf{p}_{-i}$, we can easily obtain the unique and optimal power allocation of user i, which is summarized in Theorem 1 as follows.

Theorem 1: For any fixed and feasible value of $\mathbf{p}_{-i}$, the optimal power allocation of user i, $\mathbf{p}_i^* = [P_{i,1}^*, P_{i,2}^* \cdots P_{i,N}^*]^T$, in any SINR regions is unique and given by

$$\begin{aligned}P_{i,k}^* = \Bigg[& \frac{-\left(|g_{\mathcal{S},i}^k|^2 P_{\mathcal{S},k} + 2N_0\right)\Delta_{i,k}}{2|h_{i,i}^k|^2 N_0} \\ & + \frac{\sqrt{|g_{\mathcal{S},i}^k|^4 P_{\mathcal{S},k}^2 \Delta_{i,k}^2 + 4\lambda_i |g_{\mathcal{S},i}^k|^2 |h_{i,i}^k|^2 P_{\mathcal{S},k} N_0 \Delta_{i,k}}}{2|h_{i,i}^k|^2 N_0}\Bigg]_0^{P_{i,k}^{\max}}\end{aligned} \tag{9}$$

where $[x]_a^b = \max\{a, \min\{x,b\}\}$, $\Delta_{i,k} = \sum_{j=1,j\neq i}^{Q} |h_{j,i}^k|^2 P_{j,k} + N_0$, for $k = 1, 2 \cdots N$, and λ_i is a constant chosen to satisfy the power constraint $\sum_{k=1}^{N} P_{i,k} = P_i^{\max}$.

Proof: Using standard optimization techniques, the solution can be obtained from the KKT conditions [29]. The details are omitted due to the limited space. ■

For the convenience of notation, we express the power allocation strategy of each user i as follows

$$\mathbf{p}_i^* = \mathrm{PowerAlloc}(\mathbf{p}_{-i}). \tag{10}$$

Then, based on Theorem 1, all the Nash equilibria, if they exist, must satisfy the following set of nonlinear equations:

$$P_{i,k}^* = \left[\mathrm{PowerAlloc}(\mathbf{p}_{-i}^*)\right]_k, \quad \forall\, k = 1, 2 \cdots N,\ \forall\, i \in \Omega,. \tag{11}$$

With regard to the existence of NE in the game $\mathcal{G}$, we have the following proposition which guarantees that the solution set of the fixed-point equations in (11) is always nonempty.

Proposition 1: The game $\mathcal{G}$ admits at least one NE regardless of the channel gains.

Proof: It can be easily shown that the set $\mathcal{P}_i$ of all feasible power allocation strategies of user i, as defined in (6), is convex and compact, for all $i \in \Omega$. Furthermore, the payoff function of each user i, i.e., $R_i(\mathbf{p}_i;\mathbf{p}_{-i})$, is continuous in $\mathbf{p} = (\mathbf{p}_i;\mathbf{p}_{-i})$ and strictly concave in $\mathbf{p}_i \in \mathcal{P}_i$ given any strategy $\mathbf{p}_{-i}$. Hence, the payoff function is also quasi-concave [29]. Then, based on the fundamental game theory result summarized in Proposition 20.3 in [30], we conclude that the game $\mathcal{G}$ in (5) admits at least one NE. ■

We observe that, in order to compute (9) and respond optimally to the strategy of the other users, the local information needed at user i includes $\{g_{\mathcal{S},i}^k, h_{i,i}^k, \sum_{j=1,j\neq i}^{Q} |h_{j,i}^k|^2 P_{j,k} + N_0,\ \text{for}\ k = 1, 2 \cdots N\}$, where $\sum_{j=1,j\neq i}^{Q} |h_{j,i}^k|^2 P_{j,k} + N_0$ is the sum interference plus noise, which can be measured at $\mathcal{D}_i$. Besides its local information, some global information, i.e., $P_{\mathcal{S},k}$ for $k = 1, 2 \cdots N$, also needs to be known by each user. The values of $P_{\mathcal{S},k}$, for $k = 1, 2 \cdots N$, can be obtained by either broadcasting them to all the users by the base station or calculating $P_{\mathcal{S},k} = [\frac{P_{i,k}}{(\alpha_{\mathcal{R}_i}^k)^2} - N_0]/|g_{\mathcal{S},i}|^2$. The details of how to acquire such knowledge is beyond the scope of this paper and interested readers may refer to [15] and references therein. With only local information at each user, the distributed power allocation algorithm based on the best response can be described in Algorithm I as follows.

Algorithm I: Distributed Power Allocation

Step 1: $n = 0$; **Choose a feasible** $\mathbf{p}_i^{(n)}$, **for** $i = 1, 2 \cdots Q$
Step 2: $\mathbf{p}_i^{(n+1)} = \mathrm{PowerAlloc}(\mathbf{p}_{-i}^{(n)})$, **for** $i = 1, 2 \cdots Q$;
$n = n + 1$;
Repeat Step 2 until convergence or $n = N_{\mathrm{it}}$

It is known that efficient numerical methods are available to compute the water-filling solution [16][26]. As a result of the square-root operator, however, it incurs a high computational complexity to compute the optimal power allocation given in (9). Furthermore, it is challenging to establish the sufficient conditions for the convergence of Algorithm I, though the convergence is observed in many simulations. In order to solve the power allocation problem efficiently and characterize analytically the convergence condition, we shall consider low SINR and medium to high SINR regions separately and propose another two algorithms in which each user sub-optimally allocates its power across all the subchannels depending on the channel conditions. The convergence of the algorithms is mathematically tractable and, despite the sub-optimal choice of power allocation strategy in each iteration, the performance loss in terms of the sum rate of all users is negligible, as will be shown from the extensive simulations.

B. Low SINR Regions

In this subsection, we consider the scenario in which the destination node can only receive a low SINR. This is not unusual in wireless networks when harsh transmission environments occur, e.g., randomly faded channels, severe interferences and low power constraints. In sufficiently low SINR regions, an important property of the transmission rate in (4) is that it can be appropriately approximated as a linear function of the receive SINR [28], which can be well justified by the fact that $\log(1+x) \approx x \cdot \log(e)$, where $x > 0$ is a sufficiently small number[4]. Hence, for each user $i \in \Omega$, we can reformulate the power control problem in low SINR regions as follows

$$\max_{\mathbf{p}_i} R_i(\mathbf{p}_i;\mathbf{p}_{-i}) \Leftrightarrow \max_{\mathbf{p}_i} \sum_{k=1}^{N} \gamma_{\mathcal{D}_i,k}(\mathbf{p}_i;\mathbf{p}_{-i}) \quad s.t. \quad \mathbf{p}_i \in \mathcal{P}_i \,, \tag{12}$$

where $\gamma_{\mathcal{D}_i,k}(\mathbf{p}_i;\mathbf{p}_{-i})$, as a function of the power allocation strategies of both user i and the other users, is the receive SINR of user i over the k-th subchannel given in (3) and $\mathcal{P}_i$ is the admissible power strategy of user i defined in (6). In general, the optimization problem of maximizing $\sum_{k=1}^{N} \gamma_{\mathcal{D}_i,k}(\mathbf{p}_i;\mathbf{p}_{-i})$ over the $\mathbf{p}_i$ and $\mathbf{p}_{-i}$ is a nonconvex problem and the global optimal solution is difficult to obtain. However, for each user $i \in \Omega$ in the non-cooperative game $\mathcal{G}$, it solves the optimization problem in (12) by allocating its power across all the subchannels while treating the power allocation strategy of all the other users as fixed. We can easily prove that, for a fixed $\mathbf{p}_{-i}$, maximizing $\sum_{k=1}^{N} \gamma_{\mathcal{D}_i,k}(\mathbf{p}_i;\mathbf{p}_{-i})$ over the $\mathbf{p}_i$ is a convex optimization problem and, fortunately, the closed-form solution can also be readily obtained. The following theorem summarizes the power allocation strategy of user i, given the strategy of the other users.

Theorem 2: For any fixed and feasible value of $\mathbf{p}_{-i}$, the optimal power allocation of user i, $\mathbf{p}_i^* = [P_{i,1}^*, P_{i,2}^* \cdots P_{i,N}^*]^T$, in low SINR regions is unique and given by

$$P_{i,k}^* = \left[\frac{1}{\lambda_i} \sqrt{\frac{|g_{\mathcal{S},i}^k|^2 P_{\mathcal{S},k}}{N_0} \cdot \frac{\sum_{j=1,j\neq i}^{Q} |h_{j,i}^k|^2 P_{j,k} + N_0}{\beta_{i,k}}} - \frac{\sum_{j=1,j\neq i}^{Q} |h_{j,i}^k|^2 P_{j,k} + N_0}{\beta_{i,k}} \right]_0^{P_{i,k}^{\max}} , \tag{13}$$

where λ_i is the Lagrangian multiplier chosen to satisfy the power constraint $\sum_{k=1}^{N} P_{i,k} = P_i^{\max}$ and

$$\beta_{i,k} = \frac{|h_{i,i}^k|^2 N_0}{|g_{\mathcal{S},i}^k|^2 P_{\mathcal{S},k} + N_0} \,. \tag{14}$$

Proof: The solution can be obtained by simple manipulations using the Karush-Kuhn-Tucker (KKT) conditions [29]. The details are omitted here due to space limitations. ■

[4]Without approximation, the solution becomes mathematically involved (see Eqn.9). Moreover, the proposed algorithm converges in just a small number of iterations using the approximation, which can be seen from the numerical results.

Denote $P_{i,k}^* = \left[\overline{\mathcal{WF}}_i(\mathbf{p}_{-i})\right]_k$. Then, in low SINR regions, it is clear that the NE of the game $\mathcal{G}$ satisfies

$$P_{i,k}^* = \left[\overline{\mathcal{WF}}_i(\mathbf{p}_{-i}^*)\right]_k , \quad \forall\, k = 1,2\cdots N,\ \forall\, i \in \Omega \,. \tag{15}$$

Furthermore, in low SINR regions wherein user i chooses (13) as its power allocation strategy, there exists at least one NE in the game $\mathcal{G}$, as stated in Proposition 2.

Proposition 2: In low SINR regions, the game $\mathcal{G}$ admits at least one NE regardless of the channel gains.

Proof: The proof follows the same steps as those involved in proving Proposition 1. ■

Now, we develop a distributed power allocation algorithm, which can achieve the NE based on only local information, and find the sufficient conditions under which the proposed distributed algorithm will converge to the NE regardless of the initial points. It can be seen from (13) that the optimal power allocation of user i, given the strategy of all other users, is a modified water-filling solution [17]. Specifically, the water level is modified on a per-subchannel basis by multiplying a certain factor that is a function of the power allocation strategy of the other users. Hence, we can rewrite the optimal power allocation of user i in (13) as $P_{i,k}^* = \left[\overline{\mathcal{WF}}_i(\mathbf{p}_{-i})\right]_k = \left[\frac{1}{\lambda_i t_{i,k}} - \frac{\sum_{j=1,j\neq i}^{Q} |h_{j,i}^k|^2 P_{j,k} + N_0}{\beta_{i,k}}\right]_0^{P_{i,k}^{\max}}$, for $k = 1,2\cdots N$ and $i \in \Omega$, where

$$t_{i,k} = \left(\sqrt{\frac{|g_{\mathcal{S},i}^k|^2 P_{\mathcal{S},k}}{N_0} \cdot \frac{\sum_{j=1,j\neq i}^{Q} |h_{j,i}^k|^2 P_{j,k} + N_0}{\beta_{i,k}}} \right)^{-1} . \tag{16}$$

Motivated by the idea of the modified water-filling algorithm with additional "taxation terms" [17][22], we first fix $t_{i,k}$ and solve the power allocation strategy of user i using (13) while treating the interference term caused by the other users as a constant. Then, we update $P_{i,k}$, according to (13), repeatedly until convergence or the maximum number of iteration is reached, for $i \in \Omega$ and $k = 1,2\cdots N$. After updating $\mathbf{p}$, we shall update $t_{i,k}$ based on (16), for $i \in \Omega$ and $k = 1,2\cdots N$. Specifically, the modified iterative water-filling consists of two loops. The outer loop is essentially updating $t_{i,k}$, while the inner loop is that, assuming that the multiplication factor $t_{i,k}$ is fixed, we iterate the power allocation process over all the users. Note that the water level λ_i can be efficiently determined using numerical methods [17][26], e.g., bisection, and thus the optimal power allocation of user i can be obtained for all $i \in \Omega$. Let $\mathbf{t}_i = [t_{i,1}, t_{i,2} \cdots t_{i,N}]^T$, for $i \in \Omega$, $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2 \cdots \mathbf{t}_Q)$, and N_{it} and $\hat{N}_{\text{it}}$ be the maximum numbers of iterations in the inner loop and outer loop, respectively. In general, both N_{it} and $\hat{N}_{\text{it}}$ are chosen to be sufficiently large numbers. The proposed modified iterative water-filling algorithm is then formally summarized in the following.

In general, a sufficient condition for the global convergence of Algorithm II is difficult to establish due to the two coupled update processes in Step 2 and Step 3. To be more specific, given a fixed value of $\mathbf{t}$, we can establish the convergence condition for the inner-loop iterative process based on the contraction mapping theory [25][31]. Nevertheless, the value of $\mathbf{t}$ changes over time and thus, significantly complicates

Algorithm II: Distributed Power Allocation in Low SINR Regions

Step 1: $m = n = 0$; **Choose a feasible** $\mathbf{p}_i^{(n)}$ **and** $\mathbf{t}_i^{(n)}$, **for** $i = 1, 2 \cdots Q$

Step 2: $\mathbf{p}_i^{(n+1)} = \left[\overline{\mathcal{WF}}_i(\mathbf{p}_{-i}^{(n)})\right]$ (Eqn. 13), **for** $i = 1, 2 \cdots Q$; $n = n + 1$; **Repeat Step 2 for** N_{it} **times**

Step 3: $m = m + 1$; **Update** $\mathbf{t}^{(m)}$ **until convergence or** $m = \hat{N}_{\text{it}}$; **go back to Step 2**

the derivation of conditions on the uniqueness of NE and the convergence of Algorithm II[5]. Hence, following the existing literature (e.g., [20]), we first analyze the convergence property of the inner loop in Algorithm II and then discuss the uniqueness and convergence of the whole algorithm.

Before deriving the convergence condition for the inner loop in Algorithm II that is applicable in low SINR regions, we define the matrix

$$[\mathbf{S}^{\max}]_{i,j} \triangleq \begin{cases} \max_{k=1,2\cdots N} \frac{|h_{j,i}^k|^2}{\beta_{i,k}} \sqrt{\rho_{j,i}^k}, & \text{if } i \neq j, \\ 0, & \text{otherwise}, \end{cases} \tag{17}$$

where $\beta_{i,k}$ is defined in (14) and

$$\rho_{j,i}^k = \sqrt{\frac{|g_{\mathcal{S},j}^k|^2}{|g_{\mathcal{S},i}^k|^2} \cdot \frac{\beta_{i,k}}{\beta_{j,k}} \cdot \frac{\sum_{l=1,l\neq j}^{Q} |h_{l,j}^k|^2 P_{l,k}^{\max} + N_0}{N_0}}. \tag{18}$$

Now, introducing the matrix norm $\|\cdot\|_{\infty,\text{mat}}$ induced by the vector maximum norm $\|\cdot\|_\infty$ defined as

$$\|\mathbf{A}\|_{\infty,\text{mat}} \triangleq \max_{i=\{1,2\cdots Q\}} \sum_{j=1}^{Q} [\mathbf{A}]_{i,j}, \qquad \mathbf{A} \in \mathbb{R}^{Q\times Q}, \tag{19}$$

we establish the following theorem that gives the sufficient conditions for the inner loop in Algorithm II to converge from any initial points.

Theorem 3: If the following condition is satisfied:

$$\|\mathbf{S}^{\max}\|_{\infty,\text{mat}} < 1, \tag{20}$$

where $\mathbf{S}^{\max}$ and the matrix norm $\|\cdot\|_{\infty,\text{mat}}$ are defined respectively in (17) and (19), then, as $N_{\text{it}} \to \infty$, the proposed simultaneous iterative modified water-filling power allocation algorithm in the inner loop (Step 2) in Algorithm II converges linearly to a unique point regardless of the initial feasible points.

Proof: See the appendix. ■

Given that $\mathbf{t}$ is fixed, Theorem 3 states the sufficient condition for convergence of the inner loop in Algorithm II and can be interpreted as follows. The inner loop is guaranteed to converge under weak interference conditions, i.e., for all the users, the equivalent sum interference channel gain divided by the desired channel gain is less than one. This coincides with the sufficient conditions for convergence derived for conventional Gaussian interference channels with one-hop transmission [16][25]. It is also interesting to note that the condition in (20) is independent of the values of $\mathbf{t}$ and that the inner loop may also converge even though (20) is not satisfied. As in [17]–[22], due to the time-varying nature of $\mathbf{t}$, it is mathematically intractable to derive the convergence condition for the outer loop. Fortunately, the convergence of the outer loop in Algorithm II can be guaranteed in most cases by incorporating a memory factor [17]. Specifically, according to [17], we can update $\mathbf{t}$ based on the following recursive relation

$$\mathbf{t}^{m+1} = \theta \mathbf{t}^m + (1-\theta)\hat{\mathbf{t}}^{m+1} \tag{21}$$

where $\theta \in [0, 1)$ is a constant referred to as *memory factor* and $\hat{t}_{i,k}^{m+1}$ is calculated directly from (16). It is shown in [17] that, by appropriately choosing a memory factor, the convergence of updating $\mathbf{t}$ is guaranteed at the expense of potentially slowing down the convergence speed of Algorithm II. However, we observe through extensive simulations that the convergence probability of updating $\mathbf{t}$ based on (21) is insensitive to the choice of θ. In other words, given any valid value of $\theta \in [0, 1)$, the convergence of the outer loop can be (almost) always observed. This can be explained by the fact that, when the inner loop converges under mild interference conditions, one user's power allocation strategy does not heavily rely on the others' and hence, rate maximization at each user can be viewed as "independent" [19]. Therefore, updating $\mathbf{t}_i$ by user i is more or less self-contained and the outer loop converges, given different values of θ [18]. In particular, a good choice of θ is heuristically found to be between 0 and 0.5. We further note that, if the process of updating $\mathbf{t}$ and $\mathbf{p}$ converges, Algorithm II will then converge to the NE of the game $\mathcal{G}$, since it adopts the optimal solution to the convex optimization problem (12) for each user $i \in \Omega$ [17].

In addition to convergence, the uniqueness of NE is another critical property in game-theoretic studies. Regarding the power allocation problem in this paper, however, it should be noted that the two-loop structure in Algorithm II is essentially a complicated iterative process in the sense that the parameters updated in the outer loop are time varying and highly coupled with the inner loop. As a consequence, it is quite involved to derive the condition on the uniqueness of NE of the game. The same limitation exists in [17]–[22] wherein only partial uniqueness and convergence conditions were derived by fixing the outer-loop parameters. Nevertheless, as long as the level of interferences is not high and the inner loop converges, we observe in (almost) all the cases through simulations the uniqueness and convergence of the two-loop Algorithm II[6]. Therefore, we believe that the derived partial convergence condition in (20) is still useful, since it guarantees the convergence of the inner loop and, with a large probability close to one, the convergence of the outer loop and the uniqueness of NE as well.

[5]To the best of our knowledge, the same problem was often encountered in other works as well (e.g., [17]–[22]). For instance, the authors in [18] proposed a modified iterative water-filling algorithm based on two alternate update processes (i.e., outer and inner loops in this paper) and remarked that the proposed algorithm was observed to converge to a fixed point in all the simulations, without providing rigorous analysis on the convergence and uniqueness conditions.

[6]Similar observations were reported in other works as well, e.g., [18].

C. Medium to High SINR Regions

In the previous subsection, we have developed the distributed power allocation algorithms in low SINR regions. If the channel is in a good condition, e.g., the relay node is close to the destination, or the nodes are transmitting at a high power, the assumption of low SINR will become invalid and Algorithm II is no longer suitable. Accordingly, we shall consider in this subsection the scenario in which medium to high SINR can be observed at the destination nodes.

Before developing our distributed power allocation algorithm, it is worth noting that, by exploiting the logarithm structure of the achievable rate, several approximation techniques were proposed to facilitate the analysis (see [19] and references therein). In particular, given a single-user amplify-and-forward relay network, one of the widely-used approximation techniques is neglecting the noise term that appears in the denominator of the SNR expression, i.e., $\frac{|g|^2|h|^2P_sP_r}{|g|^2P_s+|h|^2P_r} \approx \frac{|g|^2|h|^2P_sP_r}{|g|^2P_s+|h|^2P_r+N_0}$ [23]. The approximation has been shown to be sufficiently tight, especially in high SNR regions [23]. In this paper, if the $\mathcal{S}-\mathcal{R}_i$ channel is in a good condition, the Gaussian noise forwarded by $\mathcal{R}_i$ is negligible, compared to the desired signal component, and thus we can readily approximate the signal received by $\mathcal{D}_i$ by removing part of the propagated noise, i.e., $h_{i,i}^k\alpha_{\mathcal{R}_i}^k n_{\mathcal{R}_i}^k$. After some mathematical manipulations, the term $|h_{i,i}^k|^2N_0P_{i,k}$ is eliminated, without affecting too much the value of the receive SINR[7], from the denominator in (3) and thus, we obtain the approximated receive SINR at $\mathcal{D}_i$ over the k-th subchannel as

$$\hat{\gamma}_{\mathcal{D}_i,k} = \frac{|g_{\mathcal{S},i}^k|^2|h_{i,i}^k|^2P_{\mathcal{S},k}P_{i,k}}{\left(|g_{\mathcal{S},i}^k|^2P_{\mathcal{S},k}+N_0\right)\cdot\Delta_{i,k}}, \tag{22}$$

where $\Delta_{i,k} = \sum_{j=1,j\neq i}^{Q}|h_{j,i}^k|^2P_{j,k}+N_0$. As a result, the power allocation problem, for each user i, can be formulated in the same form as (8), except that the receive SINR in (8) is replaced by (22). Note that the approximated receive SINR in (22), which is also a upper bound on (3), can be viewed as the SINR in a classic interference channel without the relaying node [16]. Specifically, by denoting, for $k=1,2\cdots N$,

$$|f_{i,j}^k|^2 = \begin{cases} \frac{|g_{\mathcal{S},i}^k|^2P_{\mathcal{S},k}}{|g_{\mathcal{S},i}^k|^2P_{\mathcal{S},k}+N_0}\cdot|h_{i,i}^k|^2, & \text{if } i=j\in\Omega,\\ |h_{i,j}^k|^2, & \text{otherwise,}\end{cases} \tag{23}$$

we can rewrite the approximated SINR in (22) as

$$\hat{\gamma}_{i,k} = \frac{|f_{i,i}^k|^2P_{i,k}}{\sum_{j=1,j\neq i}^{Q}|f_{j,i}^k|^2P_{j,k}+N_0}. \tag{24}$$

By removing the base station from the system illustrated in Fig. 1, we may view the cellular relay network equivalently as a *virtual* ad hoc network which is the same as the classic model that have been considered in a rich body of literature, e.g., [16][24]. In the virtual network, the information bits are originally generated from the relay nodes. Then, assuming that the channel gain between $\mathcal{R}_i$ and $\mathcal{D}_j$ over the k-th subchannel is given by $|f_{i,j}^k|^2$ in (23), the power allocation problem reduces to the distributed spectral management in a Gaussian interference channel [16] which has been increasingly popular recently. It is well known that the optimal power allocation of user i, which treats the the interfering power from the other users as additive noise, is the water-filling solution [16] given by

$$\begin{aligned} P_{i,k}^* &= [\mathcal{WF}_i(\mathbf{p}_{-i})]_k \\ &= \left[\frac{1}{\lambda_i} - \frac{\sum_{j=1,j\neq i}^{Q}|f_{j,i}^k|^2P_{j,k}+N_0}{|f_{i,i}^k|^2}\right]_0^{P_{i,k}^{\max}} \end{aligned} \tag{25}$$

where $|f_{i,j}^k|^2$ is given in (23) and λ_i is chosen to satisfy the total power constraint of user i, for $i=\Omega$ and $k=1,2\cdots N$. Based on the approximated SINR in (22), there also exists at least one NE in the game $\mathcal{G}$ which can be easily proved following the proof of Proposition 1. Moreover, the NE can be reached by using the distributed algorithm, referred to as Algorithm III, which is similar to Algorithm I and omitted here for brevity. The only difference between Algorithm III and Algorithm I is that, in Algorithm III, the power update equation is given by (25). Clearly, the power allocation strategy based on (25) is sub-optimal for each user, given the strategy of the other users. Nevertheless, (25) is easier to compute compared with (9) and thus, is more applicable in real-time applications, e.g., video delivery.

Define the matrix

$$\left[\hat{\mathbf{S}}^{\max}\right]_{i,j} \triangleq \begin{cases}\max_{k=1,2\cdots N}\frac{|f_{j,i}^k|^2}{|f_{i,i}^k|^2}, & \text{if } i\neq j,\\ 0, & \text{otherwise,}\end{cases} \tag{26}$$

where $|f_{i,j}^k|^2$ is given in (23). Then, the convergence of Algorithm III is guaranteed by the following theorem, which is obtained by substituting the appropriate parameters into Theorem 1 in [25], where more sufficient conditions for the convergence can also be found.

Theorem 4: Algorithm III converges linearly to the unique NE of $\mathcal{G}$ based on the approximated receive SINR in (22), regardless of the initial points, if the following condition is satisfied:

$$\|\hat{\mathbf{S}}^{\max}\|_{\infty,\text{mat}} < 1, \quad \forall k\in\{1,2\cdots N\} \tag{27}$$

where the norm $\|\cdot\|_{\infty,\text{mat}}$ and the matrix $\hat{\mathbf{S}}^{\max}$ are defined in (19) and (26), respectively.

D. General Network Topology

In the above analysis, the low SINR and medium to high SINR regions were discussed separately for the convenience of presentation. In fact, these two scenarios can also be jointly considered. Specifically, given a general network topology wherein some users have good channel conditions (i.e., operating in medium to high SINR regions) while others experience poor channels (i.e., operating in low SINR regions), we propose Algorithm IV which also has a two-loop structure and is quite similar to Algorithm II. The description of Algorithm IV is omitted in this paper due to space limitations. In the inner loop of Algorithm IV, the users operating in low SINR regions still apply (13) to adjust their powers, while the users with good channel conditions update their powers following (25). In the outer loop, the same rule in (21) specifying the update of $\mathbf{t}$ applies to the low SINR users. However, the vector of $\mathbf{t}_i$, if user i operates in medium or high SINR regions, is

[7]Simulation results are provided in Fig. 8 to validate the approximation.

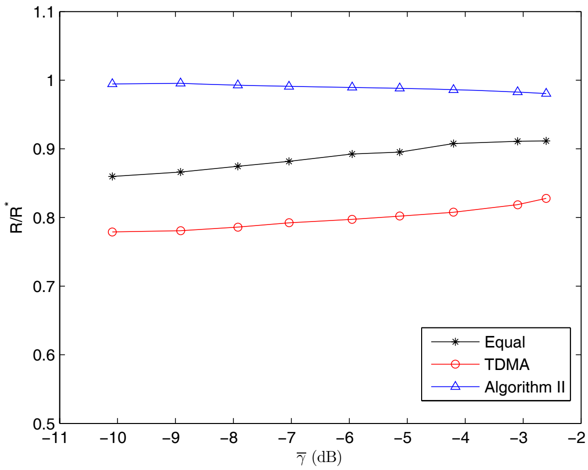


Fig. 2. Performance comparison in low SINR regions. $d_{1,2}=2$, $\theta=0.5$, $Q=2$, $K=2$, $P_{S,i}^{k}=1$, $P_i^{\max}=1$ and $P_{i,k}^{\max}=0.7$, for $i=1,2$ and $k=1,2$. R^* is the global optimal average sum rate and R is the average sum rage achieved by the proposed algorithm.

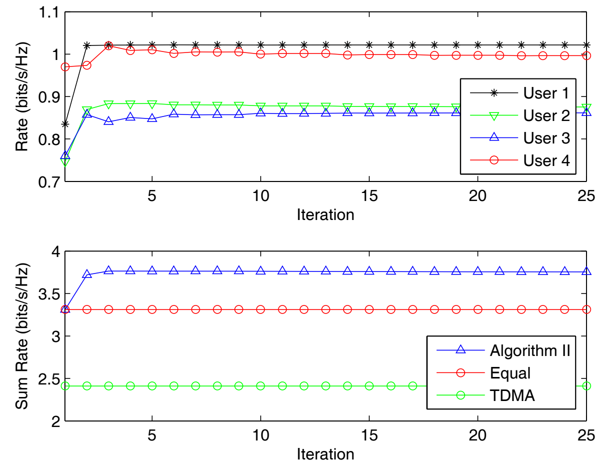


Fig. 3. Illustration of convergence in low SINR regions. Channel gains are randomly generated. θ=0.5, $Q=4$, $K=8$, $P_{S,i}^{k}=1$, $P_i^{\max}=1$ and $P_{i,k}^{\max}=0.5$, for $i=1,2\cdots4$ and $k=1,2\cdots8$.

fixed to be an identity vector, i.e., $\mathbf{t}_i=\mathbf{I}=[1,1\cdots1]$, since user i use the *un-weighted* water-filling algorithm in (25) as its power allocation. Another difference between Algorithm II and Algorithm IV is the convergence condition of the inner loop. In particular, if the conditions in Theorem 2 and Theorem 4 are satisfied for users in low SINR regions and users in medium or high SINR regions, respectively, then the inner loop of Algorithm IV converges linearly to a unique fixed point, regardless of the initial points.

IV. Numerical Results

In the simulations, we adopt the path loss fading model with Rayleigh fading and the path loss factor is set to be four. For the convenience of illustration and to limit the number of free parameters, we normalize the $\mathcal{S}-\mathcal{R}_i$ and $\mathcal{R}_i-\mathcal{D}_i$ distances to one, and assume that the *inter-user* distance between $\mathcal{R}_i$ and $\mathcal{D}_j$, denoted by $d_{i,j}$, is the same for all $i,j\in\Omega$ and $i\neq j$, as in [25][8]. It should be noted that the proposed algorithms can also be applied to any other network topologies with any number of subchannels. To reflect the channel conditions, we denote $\bar{\gamma}$ as an indicator which is defined as the average receive SINR in one sub-channel at a particular user under the equal power allocation scheme.

A. Low SINR Regions

We first characterize in Fig. 2 the performance gap in terms of the average sum rate between Algorithm II and the global optimum. Since the non-convex sum-rate maximization problem is "difficult" in the sense that no efficient algorithms exist to compute the global optimum, we focus on the two-user two-subchannel case for illustration purposes. In general, NE is not a Pareto-optimal operating point in the considered non-cooperative game. However, Algorithm II yields an average sum rate relatively close to the global optimum under the condition of mild interferences, as shown in Fig. 2. The

[8]We choose such a topology only for the convenience of illustration.

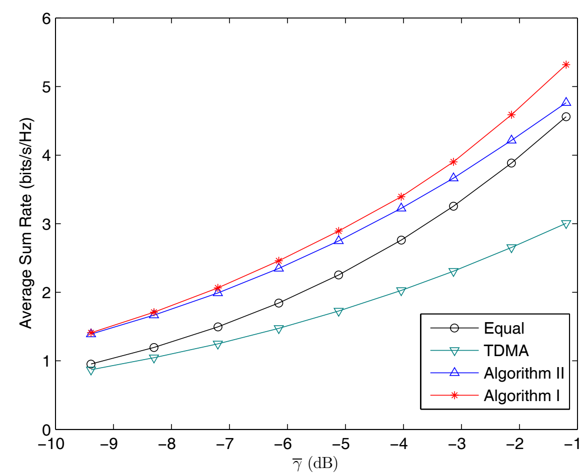
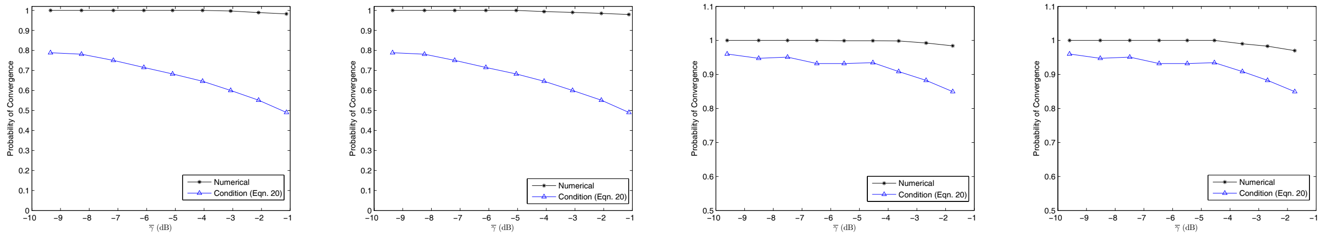


Fig. 4. Performance comparison in low SINR regions. $d_{1,2}=4$, θ=0.5, $Q=4$, $N=8$, $P_{S,i}^{k}=1$, $P_i^{\max}=1$ and $P_{i,k}^{\max}=0.5$, for $i=1,2\cdots4$ and $k=1,2\cdots8$.

same observation is also made in the conventional one-hop interference channels, e.g., [19]. Then, we illustrate in Fig. 3 the convergence of Algorithm II and show the average sum rate achieved by different schemes under different channel conditions in Fig. 4. By applying the proposed iterative power allocation algorithm with a moderate level of interference, the system-wide performance in terms of the average sum rate can be improved as opposed to the equal power allocation strategy and the conventional TDMA protocols, even though the users behave non-cooperatively to maximize their own rates at each iteration. Fig. 4 also shows that the performance gap between the best response based Algorithm I and the approximation based Algorithm II is negligible (especially when users suffer from poor channel conditions), which justifies the approximation technique used in low SINR regions. It can be seen from Fig. 5 that, although the derived analytical sufficient condition (20) in Theorem 2 is not satisfied, the convergence of Algorithm II can be observed numerically in almost all



(a) $Q = 4,\ N = 8,\ \theta = 0.5$. (b) $Q = 4,\ N = 8,\ \theta = 0.1$. (c) $Q = 2,\ N = 6,\ \theta = 0.5$. (d) $Q = 2,\ N = 6,\ \theta = 0.1$.

Fig. 5. Probability of convergence in low SINR regions. $P_{S,i}^k = 1$, $P_i^{\max} = 1$ and $P_{i,k}^{\max} = 0.5$, for $i = 1, 2 \cdots Q$ and $k = 1, 2 \cdots N$.

the channel realizations when the users are not located very close to each other. Furthermore, the probability that (20) is satisfied is independent of the value of θ and the convergence probability of Algorithm II is almost not affected by the choice of the memory factor θ. Nonetheless, we note that, as in the case of single-hop Gaussian interference channels [16], the proposed iterative algorithm may not converge when the inter-user distance is sufficiently small, or equivalently the interference level is strong and the system becomes interference-limited. Next, we show in Fig. 6 the performance comparison in terms of the average sum rate versus the number of users in the system. In the conventional TDMA protocol, the average sum rate is a constant regardless of the number of users in the system, since the base station always transmits for a fraction of 1/2 of each frame, i.e., it transmits for a fraction of $1/2Q$ to each user. In contrast, the average sum rate increases, when there are more users in the system[9], by spatially reusing the relaying slot and effectively combating the effects of interferences using Algorithm II. The underlying reason is that, when the number of users increases, the fraction of each frame devoted to the relaying slot, i.e., $1/(Q + 1)$, becomes smaller and hence, a higher spectral efficiency can be achieved than that in the TDMA protocol. On the other hand, by applying the proposed iterative power allocation algorithm, the effect of interferences due to the relays' simultaneous transmissions can then be effectively reduced and thus, the average sum rate is significantly improved compared to the equal power allocation algorithm. Moreover, as shown in Fig. 6, Algorithm II only incurs an insignificant performance loss compared with Algorithm I. Now, using the average number of iterations required to converge as the metric, we characterize the convergence time of Algorithm II under different values of θ. It can be observed from Fig. 7 that the memory factor θ slows down the convergence speed. However, since the convergence probability is insensitive to the value of θ, a smaller value of θ is desired and in some cases, θ can even be set as zero [18], i.e., no memory in the update of $\mathbf{t}$.

B. Medium to High SINR Regions and General Network Topology

In medium to high SINR regions, similar observations can be made as those in low SINR regions. However, Algorithm I and Algorithm III can achieve only a slightly higher rate than

[9]The number of users in the system cannot be arbitrarily large. Otherwise, the sum interference becomes intolerable and the average sum rate may even decrease, as in the case of single-hop interference-limited networks without interference cancelation.

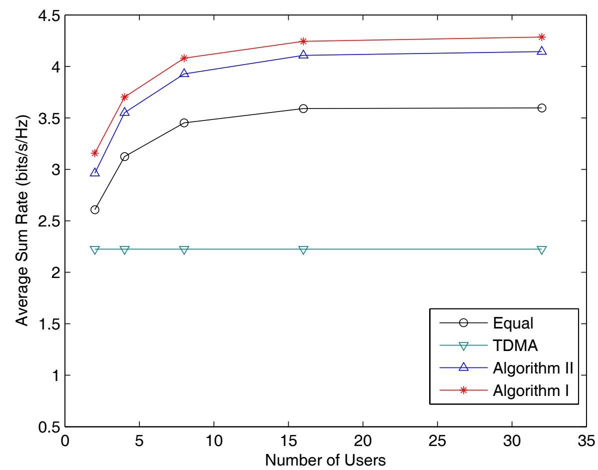
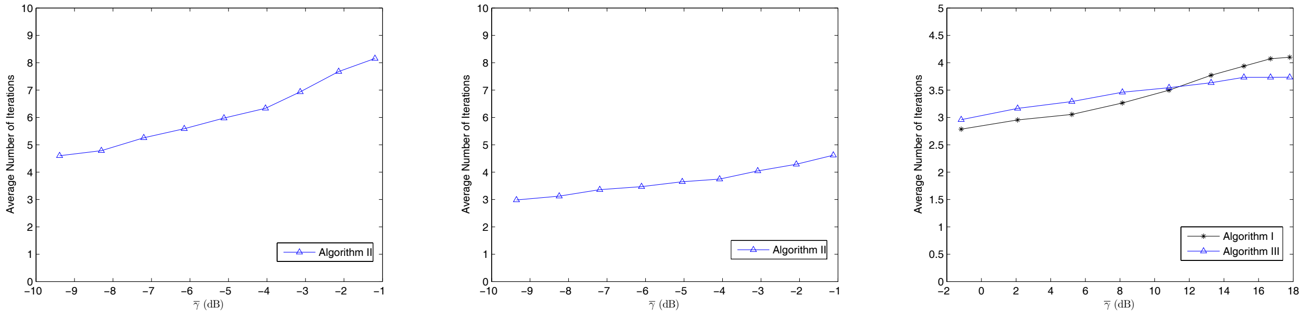


Fig. 6. Performance comparison in low SINR regions. $\bar{\gamma} = -3.5$dB, θ=0.5, $K = 8$, $P_{S,i}^k = 1$, $P_i^{\max} = 1$ and $P_{i,k}^{\max} = 0.5$, for $i = 1, 2 \cdots Q$ and $k = 1, 2 \cdots 8$.

the equal power allocation scheme. This is because the water-filling algorithm works well only in low SINR regions and the performance gain in medium to high SINR regions vanishes [13]. This phenomena can also be explained mathematically by the fact that $x_1 = x_2 = \cdots = x_M = \frac{1}{M}$ is a close-to-optimal solution to the problem $\max_{\mathbf{x}} \sum_{i=1}^{M} \log(1 + a_i x_i)$ subject to $\mathbf{x} \succeq \mathbf{0}$ and $\sum_{i=1}^{M} x_i = \bar{x}$, where $\bar{x}$ is a sufficiently large number. Due to space limitations, we only illustrate in this paper the convergence time of the proposed algorithms and the performance comparison in terms of the average sum rate under different channel conditions. Specifically, in Fig. 8, we show that the proposed Algorithm III based on the sub-optimal response can achieve almost the same average sum rate as Algorithm I, which validates the use of the approximated SINR in (22). In Fig. 7(c), we compare Algorithm I and Algorithm III from the perspective of convergence time. It shows that, on average, these two algorithms require nearly the same number of iterations to converge. Therefore, the approximation in (24) does not result in a significant performance loss or increase in the convergence time. Finally, Fig. 9 shows that, given a general network topology wherein users experience heterogeneous channel conditions, Algorithm IV achieves a higher average sum rate than equal power allocation algorithm and the conventional TDMA approach, while incurring only a slight performance loss compared to Algorithm I.



(a) $Q = 4,\ N = 8,\ \theta = 0.5$. (b) $Q = 4,\ N = 8,\ \theta = 0.1$. (c) $Q = 4,\ N = 8$.

Fig. 7. Average number of iterations required to converge. $P_{S,i}^{k} = 1$, $P_i^{\max} = 1$ and $P_{i,k}^{\max} = 0.5$, for $i = 1, 2 \cdots 4$ and $k = 1, 2 \cdots N$.

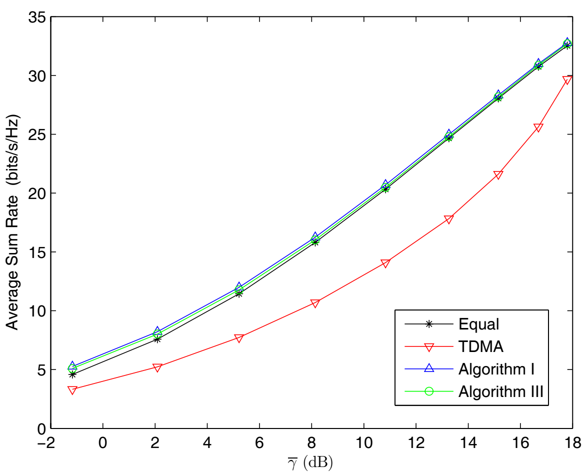


Fig. 8. Performance comparison in medium to high SINR regions. $d_{1,2} = 4$, $Q = 4$, $N = 8$, $P_{S,i}^{k} = 1$, $P_i^{\max} = 1$ and $P_{i,k}^{\max} = 0.5$, for $i = 1, 2 \cdots 4$ and $k = 1, 2 \cdots 8$.

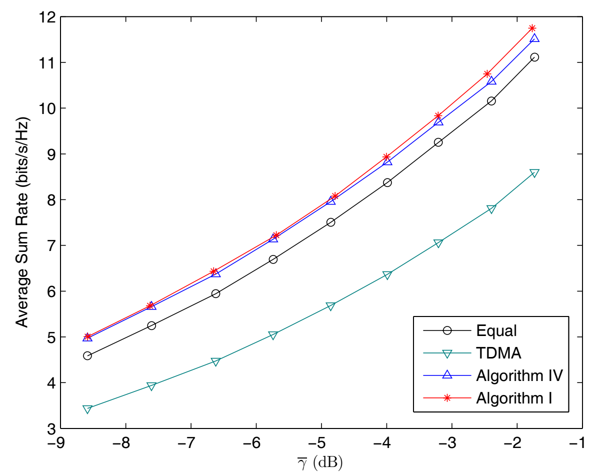


Fig. 9. Performance comparison in general network topologies. $d_{1,2} = 4$, θ=0.5, $Q = 4$, $N = 8$, $P_{S,i}^{k} = 1$, $P_i^{\max} = 1$ and $P_{i,k}^{\max} = 0.5$, for $i = 1, 2$ and $k = 1, 2 \cdots 8$ (low SINR). $P_{S,i}^{k} = 1$, $P_i^{\max} = 100$ and $P_{i,k}^{\max} = 50$, for $i = 3, 4$ and $k = 1, 2 \cdots 8$ (medium to high SINR). $\bar{\gamma}$ is the average receive SINR in one sub-channel at user 1 under the equal power allocation scheme.

Before concluding this section, we note that if the inter-user distance is sufficiently small, the interference will become a dominant factor in the receive SINR [5][16][19] and thus, it will limit the performance of the proposed algorithms. In this case, the conventional TDMA protocol may outperform the proposed algorithms at the expense of a higher complexity in coordinating the relays' transmissions, which is difficult to implement in some systems, wherein there are no message exchanges or coordination possible between users.

V. Conclusion

In this paper, we considered the downlink transmission in a wireless multi-user multi-channel cellular relay network, wherein the users have no access to the global CSI, and derived distributed power allocation algorithms using the framework of game theory. In contrast with conventional relay networks, simultaneous transmissions from different relays are allowed. The proposed power allocation algorithms can efficiently combat the interferences caused at the destination and provide a significant gain in terms of the average sum rate based solely on the local information. First, we developed a distributed algorithm based on the best response which is applicable in any SINR regions but mathematically involved to compute and analyze. Then, we considered low SINR regions and proposed an iterative modified water-filling algorithm. Specifically, each relay updates its power allocation by multiplying the water level by a factor on the per-subchannel basis. The existence of NE along with the sufficient condition to reach a NE was also determined. Next, by focusing on medium to high SINR regions, we proposed distributed algorithms based on the sub-optimal response. The algorithm can be equivalently viewed as the classic Gaussian interference channel model for which analytical sufficient conditions for the convergence to the unique NE can be readily obtained. Finally, we extended the analysis to a general network topology, wherein users with different channel conditions coexist, and conducted extensive simulations to verify the analysis.

Appendix

Before proving Theorem 3, we first introduce the following lemma the proof of which was given in [25].

Lemma 1: The modified water-filling solution $\mathbf{p}_i^* = \left[\overline{\mathcal{WF}}_i(\mathbf{p}_{-i})\right]$ in (13) can be expressed as the projection of $[-\mathbf{insr}_i(\mathbf{p}_{-i})]$ onto the simplex $\mathcal{S}$ defined as $\mathcal{P}_i$ in (6) with respect to the weighted Euclidean norm[10] with weights $\mathbf{t}_i = [t_{i,1}, t_{i,2} \cdots t_{i,N}]$, i.e.,

$$\mathbf{p}_i^* = \left[\overline{\mathcal{WF}}_i(\mathbf{p}_{-i})\right] = [-\mathbf{insr}_i(\mathbf{p}_{-i})]_{\mathcal{P}_i}^{\mathbf{t}_i} \tag{28}$$

where $[-\mathbf{insr}_i(\mathbf{p}_{-i})]_k \triangleq -\frac{\sum_{j=1,j\neq i}^{Q} |h_{j,i}^k|^2 P_{j,k}+N_0}{\beta_{i,k}}$, for $k = 1, 2 \cdots N$. ■

We now give the main properties of the modified water-filling projection operator in (28) that will be fundamental to find the sufficient conditions for the convergence of the inner loop in Algorithm II. For notational convenience, we rewrite the projection operator, for each user i, as $\mathbf{p}_i^* = \left[\overline{\mathcal{WF}}_i(\mathbf{p}_{-i})\right] = \left[-\mathbf{n}_{0,i} - \sum_{j=1,j\neq i}^{Q} \mathbf{H}_{j,i}\mathbf{p}_j\right]_{\mathcal{P}_i}^{\mathbf{t}_i}$, where $\mathbf{n}_{0,i} \triangleq [N_0/\beta_{i,1}, N_0/\beta_{i,2} \cdots N_0/\beta_{i,N}]^{\mathrm{T}}$, $\mathcal{P}_i$ and $\mathbf{t}_i$ are defined in (6) and (16), respectively, and

$$\mathbf{H}_{j,i} \triangleq \mathrm{diag}\left(\frac{|h_{j,i}^1|^2}{\beta_{i,1}}, \frac{|h_{j,i}^2|^2}{\beta_{i,2}} \cdots \frac{|h_{j,i}^N|^2}{\beta_{i,N}}\right). \tag{29}$$

Furthermore, denote the admissible set of power allocation strategies of all the users by $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2 \cdots \times \mathcal{P}_Q$, and define, for each user i, the mapping $\mathbf{T}(\mathbf{p}) = (\mathbf{T}_i(\mathbf{p}))_{i\in\Omega} : \mathcal{P} \to \mathcal{P}$ as

$$\mathbf{T}_i\left(\mathbf{p}\right) \triangleq \left[-\mathbf{n}_{0,i} - \sum_{j=1,j\neq i}^{Q} \mathbf{H}_{j,i}\mathbf{p}_j\right]_{\mathcal{P}_i}^{\mathbf{t}_i}. \tag{30}$$

Following the proof of Proposition 1 that ensures the existence of NE in the game $\mathcal{G}$ in low SINR regions, we can easily show that, given a fix value of $\mathbf{t}$, there exists at least one equilibrium point in the iteration process of the inner loop in Algorithm II, which guarantees that the solution set to the following set of equations is always nonempty: $\mathbf{p}_i^* = \mathbf{T}_i\left(\mathbf{p}^*\right) = \left[-\mathbf{n}_{0,i} - \sum_{j=1,j\neq i}^{Q} \mathbf{H}_{j,i}\mathbf{p}_j^*\right]_{\mathcal{P}_i}^{\mathbf{t}_i}, \forall\, i \in \Omega$.

In [25], the authors showed that the conventional water-filling projection operator onto a convex set satisfies the non-expansive property [31]. Now, we formally extend the non-expansive property to the case of modified water-filling projection defined in (28), and summarize it as follows in Lemma 2, whose proof is omitted here for brevity.

Lemma 2: Given $\mathcal{P}_i$ in (6) and the weighted Euclidean norm defined as $\|\mathbf{x}\|_{2,\mathbf{w}} \triangleq \left(\sum_i^N w_i |x_i|^2\right)^{1/2}$ in which $\mathbf{w} = [w_1, w_2 \cdots w_N]^T$ is the corresponding weighting vector, we denote $[\,\cdot\,]_{\mathcal{P}_i}^{\|\cdot\|_{2,\mathbf{t_i}}}$ as the projection operator onto $\mathcal{P}_i$ with respect to the vector norm $\|\cdot\|_{2,\mathbf{t_i}}$, and $[\,\cdot\,]_{\mathcal{P}_i}^{\|\cdot\|_{2,\mathbf{t_i}}}$ is non-expansive, i.e., $\left\|[\mathbf{x}]_{\mathcal{P}_i}^{\|\cdot\|_{2,\mathbf{t_i}}} - [\mathbf{y}]_{\mathcal{P}_i}^{\|\cdot\|_{2,\mathbf{t_i}}}\right\|_{2,\mathbf{t_i}} \leq \|\mathbf{x}-\mathbf{y}\|_{2,\mathbf{t_i}}, \forall\, \mathbf{x}, \mathbf{y} \in \mathbb{R}_+^N$. ■

To establish the sufficient conditions under which the mapping $\mathbf{T}$ defined in (30) is block-contraction, we shall rigourously prove Proposition 3 in the following.

Proposition 3: If $\|\mathbf{S}^{\max}\|_{\infty,\mathrm{mat}} < 1$, where $\mathbf{S}^{\max}$ and the matrix norm $\|\cdot\|_{\infty,\mathrm{mat}}$ are defined in (17) and (19), respectively, then the mapping $\mathbf{T}$ defined in (30) is block-contraction with a modulus $\delta = \|\mathbf{S}^{\max}\|_{\infty,\mathrm{mat}}$, with respect to the block-maximum norm $\|\cdot\|_{\mathbf{t},\mathrm{block}}$ defined as

$$\|\mathbf{T}(\mathbf{p})\|_{\mathbf{t},\mathrm{block}} \triangleq \max_{i\in\Omega} \|\mathbf{T}_i(\mathbf{p})\|_{2,\mathbf{t_i}}\,, \tag{31}$$

where $\|\mathbf{T}_i(\mathbf{p})\|_{2,\mathbf{t_i}} = \left(\sum_{k=1}^N t_{i,k} |\left[\mathbf{T}_i(\mathbf{p})\right]_k|^2\right)^{1/2}$.

Proof: We need to show $\left\|\mathbf{T}(\mathbf{p}^{(1)}) - \mathbf{T}(\mathbf{p}^{(2)})\right\|_{\mathbf{t},\mathrm{block}} \leq \delta \cdot \left\|\mathbf{p}^{(1)} - \mathbf{p}^{(2)}\right\|_{\mathbf{t},\mathrm{block}}, \forall\, \mathbf{p}^{(1)}, \mathbf{p}^{(2)} \in \mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2 \cdots \times \mathcal{P}_Q$, to prove the contraction property of the mapping $\mathbf{T}$. Given any $\mathbf{p}^{(1)} = (\mathbf{p}_1^{(1)}, \mathbf{p}_2^{(1)} \cdots \mathbf{p}_Q^{(1)}) \in \mathcal{P}$ and $\mathbf{p}^{(2)} = (\mathbf{p}_1^{(2)}, \mathbf{p}_2^{(2)} \cdots \mathbf{p}_Q^{(2)}) \in \mathcal{P}$, we define respectively, for each user i, the *weighted* Euclidean distances between these two vectors and their projected vectors using (30) as $e_{\mathbf{T}_i} = \left\|\mathbf{T}_i(\mathbf{p}^{(1)}) - \mathbf{T}_i(\mathbf{p}^{(2)})\right\|_{2,\mathbf{t_i}}$ and $e_i = \left\|\mathbf{p}_i^{(1)} - \mathbf{p}_i^{(2)}\right\|_{2,\mathbf{t_i}}$. Then, we have the set of inequalities (32) shown on the top of the next page, where the first inequality is due to the non-expansive property of the modified water-filling projection operator defined in (28) and $\mathbf{H}_{j,i}$ is a diagonal matrix given in (29). From the definition of $t_{i,k}$ in (16), we have

$$\begin{aligned}\frac{t_{i,k}}{t_{j,k}} &= \sqrt{\frac{|g_{\mathcal{S},j}^k|^2}{|g_{\mathcal{S},i}^k|^2} \cdot \frac{\beta_{i,k}}{\beta_{j,k}} \cdot \frac{\sum_{l=1,l\neq j}^{Q} |h_{l,j}^k|^2 P_{l,k} + N_0}{\sum_{u=1,u\neq i}^{Q} |h_{u,i}^k|^2 P_{u,k} + N_0}} \\ &\leq \sqrt{\frac{|g_{\mathcal{S},j}^k|^2}{|g_{\mathcal{S},i}^k|^2} \cdot \frac{\beta_{i,k}}{\beta_{j,k}} \cdot \frac{\sum_{l=1,l\neq j}^{Q} |h_{l,j}^k|^2 P_{l,k}^{\max} + N_0}{N_0}} \\ &= \rho_{j,i}^k\,,\end{aligned} \tag{33}$$

which is independent of $P_{i,k}$, for any $i = 1, 2, \cdots Q$ and $k = 1, 2 \cdots N$. Hence, following the set of inequalities in (32), we can further derive

$$\begin{aligned} e_{\mathbf{T}_i} &\leq \sum_{j=1,j\neq i}^{Q} \max_{k=\{1,2\cdots N\}} \left([\mathbf{H}_{j,i}]_{k,k} \sqrt{\frac{t_{i,k}}{t_{j,k}}}\right) e_j \\ &\leq \sum_{j=1,j\neq i}^{Q} \max_{k=\{1,2\cdots N\}} \left([\mathbf{H}_{j,i}]_{k,k} \sqrt{\rho_{j,i}^k}\right) e_j\,.\end{aligned} \tag{34}$$

Define the vectors $\mathbf{e_T} \triangleq [e_{\mathbf{T}_1}, e_{\mathbf{T}_2} \cdots e_{\mathbf{T}_Q}]^T$ and $\mathbf{e} \triangleq [e_1, e_2 \cdots e_Q]^T$, and thus the inequality in (34) can be expressed in the vector form as $\mathbf{0} \leq \mathbf{e_T} \leq \mathbf{S}^{\max}\mathbf{e}$, where $\mathbf{S}^{\max}$ is defined in (17). Then, by applying the infinity norm $\|\cdot\|_\infty$, we obtain the following

$$\|\mathbf{e_T}\|_\infty \leq \|\mathbf{S}^{\max}\mathbf{e}\|_\infty \leq \|\mathbf{S}^{\max}\|_{\infty,\mathrm{mat}} \|\mathbf{e}\|_\infty\,, \tag{35}$$

where $\|\cdot\|_{\infty,\mathrm{mat}}$, defined in (19), is the matrix norm induced by the vector norm $\|\cdot\|_\infty$. Finally, based on (31) and (35), the following can be established:

$$\begin{aligned} &\left\|\mathbf{T}(\mathbf{p}^{(1)}) - \mathbf{T}(\mathbf{p}^{(2)})\right\|_{\mathbf{t},\mathrm{block}} \\ &= \max_{i\in\Omega} \left\|\mathbf{T}_i(\mathbf{p}^{(1)}) - \mathbf{T}(\mathbf{p}_i^{(2)})\right\|_{2,\mathbf{t}_i} \\ &= \|\mathbf{e_T}\|_\infty \leq \|\mathbf{S}^{\max}\|_{\infty,\mathrm{mat}} \|\mathbf{e}\|_\infty \\ &= \|\mathbf{S}^{\max}\|_{\infty,\mathrm{mat}} \left\|\mathbf{p}^{(1)} - \mathbf{p}^{(2)}\right\|_{\mathbf{t},\mathrm{block}}\,,\end{aligned} \tag{36}$$

[10] The weighted Euclidean norm with weights $\mathbf{w} = [w_1, w_2 \cdots w_N]$ is defined as $||\mathbf{x}||_{2,\mathbf{w}} \triangleq \left(\sum_i^N w_i |x_i|^2\right)^{1/2}$ [32].

$$
\begin{aligned}
\mathbf{e}_{\mathbf{T}_i} &= \left\| \left[-\mathbf{n}_{0,i} - \sum_{j=1,j\neq i}^{Q} \mathbf{H}_{j,i}\mathbf{p}_j^{(1)} \right]_{\mathcal{P}_i}^{\mathbf{t}_i} - \left[-\mathbf{n}_{0,i} - \sum_{j=1,j\neq i}^{Q} \mathbf{H}_{j,i}\mathbf{p}_j^{(2)} \right]_{\mathcal{P}_i}^{\mathbf{t}_i} \right\|_{2,\mathbf{t_i}} \leq \left\| \sum_{j=1,j\neq i}^{Q} \mathbf{H}_{j,i}\mathbf{p}_j^{(1)} - \sum_{j=1,j\neq i}^{Q} \mathbf{H}_{j,i}\mathbf{p}_j^{(2)} \right\|_{2,\mathbf{t_i}} \\
&= \left\| \sum_{j=1,j\neq i}^{Q} \mathbf{H}_{j,i} \left(\mathbf{p}_j^{(1)} - \mathbf{p}_j^{(2)} \right) \right\|_{2,\mathbf{t_i}} \leq \sum_{j=1,j\neq i}^{Q} \left\| \mathbf{H}_{j,i} \left(\mathbf{p}_j^{(1)} - \mathbf{p}_j^{(2)} \right) \right\|_{2,\mathbf{t_i}} = \sum_{j=1,j\neq i}^{Q} \sqrt{ \sum_{k=1}^{N} t_{i,k} \left([\mathbf{H}_{j,i}]_{k,k} \right)^2 \left[P_{j,k}^{(1)} - P_{j,k}^{(2)} \right]^2 } \\
&= \sum_{j=1,j\neq i}^{Q} \sqrt{ \sum_{k=1}^{N} t_{j,k} \left([\mathbf{H}_{j,i}]_{k,k} \sqrt{\frac{t_{i,k}}{t_{j,k}}} \right)^2 \left[P_{j,k}^{(1)} - P_{j,k}^{(2)} \right]^2 } \leq \sum_{j=1,j\neq i}^{Q} \max_{k=\{1,2\cdots N\}} \left([\mathbf{H}_{j,i}]_{k,k} \sqrt{\frac{t_{i,k}}{t_{j,k}}} \right) \sqrt{ \sum_{k=1}^{N} t_{j,k} \left[P_{j,k}^{(1)} - P_{j,k}^{(2)} \right]^2 } \\
&= \sum_{j=1,j\neq i}^{Q} \max_{k=\{1,2\cdots N\}} \left([\mathbf{H}_{j,i}]_{k,k} \sqrt{\frac{t_{i,k}}{t_{j,k}}} \right) \left\| \mathbf{p}_j^{(1)} - \mathbf{p}_j^{(2)} \right\|_{2,\mathbf{t_j}} = \sum_{j=1,j\neq i}^{Q} \max_{k=\{1,2\cdots N\}} \left([\mathbf{H}_{j,i}]_{k,k} \sqrt{\frac{t_{i,k}}{t_{j,k}}} \right) e_j
\end{aligned} \tag{32}
$$

for any $\mathbf{p}^{(1)}, \mathbf{p}^{(2)} \in \mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2 \cdots \times \mathcal{P}_Q$. Therefore, if $\|\mathbf{S}^{\max}\|_{\infty,\text{mat}} < 1$, the mapping $\mathbf{T}$ is block-contraction with the modulus $\delta = \|\mathbf{S}^{\max}\|_{\infty,\text{mat}}$. ∎

Proposition 1.1 in [31] states that, if a mapping $\mathbf{M} : \mathcal{X} \to \mathcal{X}$ is contraction with a modulus $\kappa \in [0,1)$ and $\mathcal{X}$ is a closed subset of $\mathbb{R}^n$, then the mapping $\mathbf{M}$ has a unique fixed point $x^* \in \mathcal{X}$ and, furthermore, the update sequence generated by $x(t+1) = \mathbf{M}(x(t))$ converges to the fixed point x^* given any initial value $x(0)$. In Proposition 3, we have shown that, if $\|\mathbf{S}^{\max}\|_{\infty,\text{mat}} < 1$ is satisfied, the mapping $\mathbf{T} : \mathcal{P} \to \mathcal{P}$ defined in (30) is block-contraction with a modulus $\delta = \|\mathbf{S}^{\max}\|_{\infty,\text{mat}} \in [0,1)$ with respect to the block-maximum norm defined in (31). Therefore, Theorem 3 is proved by applying Proposition 1.1 in [31] and Proposition 3.

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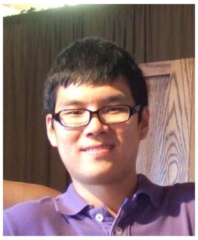

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