

# Joint Concordance Index

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**Abstract**—Existing metrics in competing risks survival analysis such as concordance and accuracy do not evaluate a model’s ability to *jointly* predict the event type and the event time. To address these limitations, we propose a new metric, which we call the joint concordance. The joint concordance measures a model’s ability to predict the overall risk profile, i.e., risk of death from different event types. We develop a consistent estimator for the new metric that accounts for the censoring bias. We use the new metric to develop a variable importance ranking approach. Using the real and synthetic data experiments, we show that models selected using the existing metrics are worse than those selected using joint concordance at jointly predicting the event type and event time. We show that the existing approaches for variable importance ranking often fail to recognize the importance of the event-specific risk factors, whereas, the proposed approach does not, since it compares risk factors based on their contribution to the prediction of the different event-types. To summarize, joint concordance is helpful for model comparisons and variable importance ranking and has the potential to impact applications such as risk-stratification and treatment planning in multimorbid populations.

## I. INTRODUCTION

### A. Motivation

The concordance index [1] is one of the most widely used metrics in survival analysis with competing risks (SA-CR) for measuring a model’s discriminative ability, i.e., a model’s ability to correctly order the subjects based on their risk. As was pointed out in [1], the concordance index is used to assess the prognostic ability of a model for one event type of interest in the presence of competing risks, but it is not adequate to assess the prognostic ability of a model when there is more than one event type of interest. Several clinical scenarios with more than one event types are given below.

#### 1) Treatment planning for multimorbid populations.

Adverse treatment reactions are one of the leading causes of death in the United States [2]. This has generated a large amount of interest to tackle the problem of polypharmacy [2]. Multi-morbidity – the accumulation of chronic diseases – has emerged as a major contemporary challenge of the aging population [3]. More than two-thirds of people aged over 65 are multimorbid, i.e., have two or more chronic diseases [4]. The current healthcare provision is not designed to consider diseases in combination leading to complications arising from unnecessary polypharmacy. Therefore, it is important to develop treatment plans in multimorbid

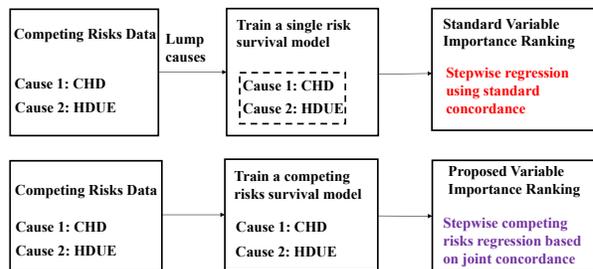


Fig. 1. Variable importance ranking for overall risk profile: standard vs proposed; CHD: Coronary Heart Disease, HDUE: Heart Disease of Uncertain Etiology

populations after assessing the overall risk profile, i.e. the risks of death from different conditions.

- 2) **Treatment planning for critical care.** SA-CR models have been used to develop early warning systems to predict the event time and the event type (e.g., ventilation or discharged alive, different types of organ failures) [5]. In these applications, the joint prediction of the event type (e.g., ventilation) and the event time is helpful for planning the allocation of resources (e.g., ventilator).

We now elaborate on why the existing metrics are not sufficient to cater to the above scenarios.

### B. Contributions

In this work, we propose a new metric that we call the *joint concordance*. The joint concordance index is the probability that a given model accurately predicts the event type for a subject while also ranking that subject’s risk correctly among the other subjects. We show that the joint concordance index can be interpreted by decomposing it into a metric that is similar to accuracy and concordance conditional on the correct predictions. We also prove that the decomposition does not lead to standard metrics, i.e., the concordance and accuracy. In addition, we prove that the joint concordance cannot be expressed as a function of the concordance and the accuracy.

In most survival analysis settings, the most common form of censoring that is observed is right censoring; right censoring occurs when either the subject is lost in the follow-up, or the study ends. In these scenarios, the estimation of the joint

concordance index can become more difficult as censoring can introduce bias in the population that is observed at different times in the follow-up. Different models can be used to estimate the censoring distribution such as the Kaplan-Meier model, the Cox model. We estimate the model of censoring based on one of these models to construct an estimator for the joint concordance index. We prove that the proposed estimator is consistent. We introduce a variable importance ranking approach that we call stepwise competing risks regression. In this approach, we train a competing risks regression model and use backward elimination, i.e. drop the variable that leads to the least change in the joint concordance (See Figure 1). We also carry out experiments on real and synthetic datasets to establish the utility of the proposed metric.

### C. Software

The code for the joint concordance index is available at <sup>1</sup>. We also developed an application, which is available at <sup>2</sup>. The input/output for application is a) Input: Upload a standard competing risks dataset and select the model: Fine-Gray or Cause-Specific Cox model. b) Output: performance of the model- concordance for each cause, accuracy and the joint concordance.

## II. EXISTING CONCORDANCE INDEX FOR COMPETING RISKS AND ITS LIMITATIONS

### A. Definition

In this section, we formally describe the most commonly used metric in SA-CR [6] for evaluating prognostic models. We begin by considering an uncensored dataset. We consider a dataset  $\mathcal{D}$  comprising of the survival (event time) data for  $n$  subjects who have been followed up for a finite amount of time. Let  $\mathcal{D} = \{X_i, T_i, D_i, i = 1, \dots, n\}$ , where  $X_i \in \mathcal{X}$  is a  $d$ -dimensional vector of covariates associated with the subject  $i$  (for instance, the information collected at baseline such as gender, age, etc.),  $T_i \in \mathbb{R}^+$  is the time until an event occurred, and  $D_i \in \mathcal{K}$  is the type of event that occurred. The set  $\mathcal{K} = \{E_1, E_2\}$  is a finite set of competing events that could occur to a subject  $i$ , where  $E_1$  is the event of the first type and  $E_2$  is the event of the second type. The Cumulative Incidence Function (CIF) [7] is the probability of occurrence of a particular event in  $d \in \mathcal{K}$  by time  $t$ , and is given by  $F_d(t|X) = Pr(T \leq t, D = d|X)$ , where  $X \in \mathcal{X}$  is the covariate vector of the subject,  $T$  is the event time, and  $D$  is the event type.

1) *Time-dependent event-specific concordance index*: The concordance index measures a model's ability to discriminate the subjects. The concordance index is defined for each event type separately. Suppose a model  $M$  predicts the risk of event  $d$  until time  $t$  to be  $M(X, t, d)$ . Consider an independent test set of i.i.d. realizations of  $(X_i, T_i, D_i)$  from the joint distribution of the covariates and the competing risks outcome. For a random pair of subjects  $(X_i, T_i, D_i)$  and  $(X_j, T_j, D_j)$ , the

time dependent concordance for the event type  $E_k$  is defined in (1). For the pair of subjects  $(X_i, T_i, D_i)$  and  $(X_j, T_j, D_j)$  in (1), the first subject  $(X_i, T_i, D_i)$  would be in greater need of treatment for the event type  $d$  than the second subject  $(X_j, T_j, D_j)$  if they experienced the event of interest ( $D_i = d$ ) and the second subject experienced the event of interest later ( $T_i < T_j$  and  $D_j = d$ ) or not at all ( $D_j \neq d$ ). The concordance vector is defined as a vector consisting of the time-dependent concordance for every event type and it is given as  $\vec{C}(t) = [C(t, 1), C(t, 2)]$ . Note that the definition trivially extends to more than two event types.

### B. Limitations of the existing concordance index

Each element of the concordance vector  $\vec{C}(t)$  defined above consists of information regarding a model's discrimination ability for each event type. However, it does not consist of information on whether the model is good at predicting the event type as well. In many applications, the evaluation of a model's ability to *jointly* predict the event type and the event time is critical. For instance, treatment planning in multimorbid populations [8], resource planning in critical care [5].

## III. JOINT CONCORDANCE INDEX

### A. Naive solution

We first describe an intuitive solution to overcome the limitations of concordance index discussed in Section II-B. Define a model  $M$ 's prediction for the event type up to time  $t$  for subject  $X_i$  as  $M_c(X_i, t)$ . We define the accuracy of a model as  $\mathcal{A}(t) = Pr(M_c(X_i, t) = D_i | T_i \leq t)$

In the definition, we condition on  $T_i \leq t$  because we can evaluate a model's prediction only for the subjects who experienced the event before the stated time horizon, i.e.,  $t$ . We construct a vector of the concordance index for all the event types, and the accuracy given as  $\mathcal{V}(t) = [C(t, 1), C(t, 2), \mathcal{A}(t)]$

Intuitively, it might seem that this vector is sufficient to capture a model's ability to make the joint prediction of the event type and event time because the accuracy contains information about the ability of a model to predict the event type and the concordance captures the ability of the model to discriminate the event time for every event type separately. This solution is appealing because it is simple, but it has limitations that we discuss next. Suppose that a model makes a correct prediction of the event type for a subject  $X_i$ . Therefore, the condition inside the probability in  $\mathcal{A}(t)$  is true for this subject. However, it is possible that for the same subject the discrimination condition inside (1) is not satisfied, which implies that the model is good at predicting the event type but not the event time for the predicted event type for this subject. Therefore, concordance and accuracy (that comprise the vector  $\mathcal{V}(t)$ ) evaluate the marginal probabilities and not the joint probabilities. The joint evaluation is not trivial as the accuracy, and the concordance events are neither independent nor imply one another.

<sup>1</sup>[https://github.com/ahujak/Concordance\\_Based\\_Metrics\\_Survival](https://github.com/ahujak/Concordance_Based_Metrics_Survival)

<sup>2</sup><https://mlinterpreter.shinyapps.io/concordance/>

$$\mathcal{C}(t, k) := Pr\left(M(X_i, t, E_k) > M(X_j, t, E_k) \mid \left[D_i = E_k \text{ and } T_i \leq t \text{ and } (T_i < T_j \text{ or } D_j \neq E_k)\right]\right) \quad (1)$$

### B. Definition

The joint concordance index is the probability that a given model accurately predicts the event type for a subject while also ranking that subject's risk correctly among the other subjects. The expression for the joint concordance is given in (2). In the equation (2), the model's ability to predict the event type for the subject and discriminate that subject from the other subjects is jointly evaluated.

### C. Relationship with the existing metrics and interpretation

In this section, we study the relationship between the joint concordance and the existing metrics. We decompose the joint concordance in equation (3) into two terms that are easier to interpret. In equation (3), the first term is concordance conditional on the event that the model correctly predicts the event type (1). The conditional concordance (3) is evaluated for the subjects for which the events are predicted correctly unlike the standard concordance (1) that is evaluated even for the subjects for which the wrong event type was predicted. The second term in the decomposition (3) is similar to the accuracy  $\mathcal{A}(t)$ . The difference between the accuracy term in (3) and  $\mathcal{A}(t)$  is that the event in the conditional probabilities is different. From (3), it might seem that the joint concordance can be expressed as a function of the existing metrics- concordance and accuracy. However, this is not the case. In the next proposition, we show that the joint concordance cannot be expressed as a function of the existing metrics defined in  $\mathcal{V}(t)$ .

**Proposition 1: Joint concordance vs. Existing metrics.**  $\exists$  no function  $f: \mathbb{R}^3 \rightarrow \mathbb{R}$  such that  $\mathcal{J}\mathcal{C}(t) = f(\mathcal{V}(t))$

The proofs to all the propositions are in the Appendix Section.

### D. Estimators of joint concordance

**Weighted estimator to account for censoring.** In the description of the dataset in Section II, we assumed that there was no censoring. We now introduce censoring variables.  $C_i$  is defined as the censoring time for subject  $i$ . For subject  $i$  we observe  $X_i, \tilde{T}_i, \tilde{D}_i, \Delta_i$ , where  $\tilde{T}_i = \min\{T_i, C_i\}$  is the event time,  $\Delta_i = I(T_i \leq C_i)$ , type of event  $\tilde{D}_i = \Delta_i D_i$ . We make the standard assumption that the censoring is independent of other variables conditional on the covariates. The probability that the subject  $i$  is uncensored up to time  $t$  is given as  $G(t) = Pr(C_i > t | X_i)$ . We use the inverse probability of censoring weighted (IPCW) (See [1]) to adjust for the bias that is introduced by censoring. We can use different models to estimate the censoring bias; we denote the estimated model of censoring as  $\hat{G}$ . The two most natural choices for estimating the censoring models are: i) Kaplan-Meier estimator and ii) Cox model estimator of the censoring distribution. We provide some notation below that is needed to define the estimator.

$\tilde{A}_{ij} = I(\tilde{T}_i < \tilde{T}_j)$ ,  $\tilde{B}_{ij}(d) = I(\tilde{T}_i > \tilde{T}_j, \tilde{D}_j \neq d)$ ,  $\tilde{N}_i(t, d) = I(\tilde{T}_i \leq t, \tilde{D}_i = d)$ ,  $\tilde{C}_{ij}(d) = I(\tilde{T}_i < \tilde{T}_j \text{ or } \tilde{D}_j \neq$

$d)$ ,  $Q_{ij}(t, d) = I(M(X_i, t, d) > M(X_j, t, d) \ \& \ M_c(X_i) = d)$   
 $\hat{W}_{ij}^1 = \frac{1}{\hat{G}(T_i|X_i)\hat{G}(T_i|X_j)}$   $\hat{W}_{ij}^2 = \frac{1}{\hat{G}(T_i|X_i)\hat{G}(T_j|X_j)}$   
 The weighted estimator is defined as

$$\hat{\mathcal{J}}\mathcal{C}_{wtd}(t) := \frac{\sum_{d \in \mathcal{K}} \sum_{i,j} (\tilde{A}_{ij} \hat{W}_{ij}^1 + \tilde{B}_{ij}(d) \hat{W}_{ij}^2) N_i(t, d) Q_{ij}(t, d)}{\sum_{d \in \mathcal{K}} \sum_{i,j} (\tilde{A}_{ij} \hat{W}_{ij}^1 + \tilde{B}_{ij}(d) \hat{W}_{ij}^2) N_i(t, d)} \quad (4)$$

The denominator in (4) counts the number of pairs in the dataset that satisfy the definition of the event in the conditional probability in (2). The numerator in (4) counts the number of jointly concordant pairs (pairs for which the prediction of the model is accurate and the concordance condition holds).

In the next proposition, we show that the weighted estimator (4) is consistent. For the next proposition, we require that the model for censoring is correctly specified. The same assumptions were also made in [6].

**Proposition 2: Properties of the estimator**  $\hat{\mathcal{J}}\mathcal{C}_{wtd}(t)$  is a consistent estimator of the joint concordance  $\mathcal{J}\mathcal{C}(t)$ .

### E. Variable importance ranking

First we highlight the limitations of the existing approaches for ranking the variables with respect to the overall risk profile and propose an alternate approach based on joint concordance that overcomes these limitations.

1) *Existing approaches:* We describe the two most common approaches that are used for variable importance ranking-standardized regression coefficients based approaches [9] and the stepwise regression based approaches [10]. The existing works [11] [12] rank the covariates for the overall risk profile by lumping the different event types into one common group and then using the standardized regression coefficients or the stepwise regression methods to rank the covariates with respect to the risk of the lumped event (See Figure 1). In the comparisons to follow, for the stepwise regression methods we use concordance index defined in (1) as the measure that is compared in each step. We also contrast our results with the standardized regression coefficient based approach.

2) *Stepwise competing risks (CR) regression approach:* We refer to the proposed approach as stepwise competing risks (CR) regression approach. First we first train a competing risks model on all the variables. We use backward elimination with stepwise regression with joint concordance as the metric. In each step of the backward elimination, we compute the joint concordance for the trained model. We drop the variable that leads to the least amount of change in the joint concordance when dropped (See Figure 1). The same procedure is repeated after dropping the variable. Note that the least important variable is dropped first and the most important variable is dropped last.

$$\mathcal{J}\mathcal{C}(t) := Pr\left(M(X_i, t, D_i) > M(X_j, t, D_i) \& M_c(X_i, t) = D_i \mid [T_i \leq t \& (T_i < T_j \text{ or } D_i \neq D_j)]\right) \quad (2)$$

$$\mathcal{J}\mathcal{C}(t) = \left( \overbrace{Pr\left(M(X_i, t, D_i) > M(X_j, t, D_i) \mid D_i = M_c(X_i, t) \& T_i \leq t \& [T_i < T_j \text{ or } D_i \neq D_j]\right)}^{\text{Concordance conditional on the correct prediction of the event type}} \right) \quad (3)$$

$$\underbrace{\left( Pr(M_c(X_i, t) = D_i \mid T_i \leq t \& [T_i < T_j \text{ or } D_i \neq D_j]) \right)}_{\text{Accuracy}^*}$$

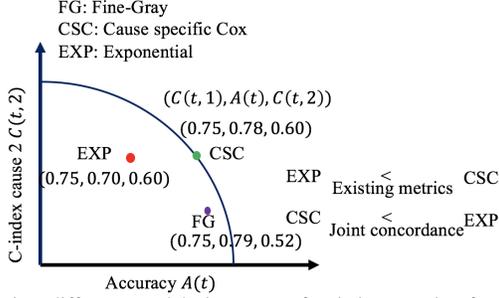


Fig. 2. Comparing different models in terms of existing metrics for the synthetic setting

#### IV. EXPERIMENTS

In this section, we first discuss the synthetic data experiments and then discuss the real data experiments. These experiments are carried out with three goals in mind:

- 1) Existing metrics are not sufficient for joint evaluation
- 2) Existing variable importance ranking often fail to rank the factors with respect to the overall risk
- 3) Compute the efficiency of the weighted estimator

All the experiments were conducted in the R.

##### A. Synthetic Data Experiments

**Synthetic experiment setting.** We use an experiment setting that is very similar to [6]. The covariate of a subject is  $X \in \mathbb{R}$ . It is drawn from a standard normal distribution. Suppose that there are three event types - event of type 1, event of type 2, and censoring. We use an accelerated failure time model [7] to model the event time. The latent time for event type  $k$  is  $T_k$  and it is drawn from an exponential distribution with arrival rate  $\lambda_k(t|X) = \lambda_k(t)\exp(\beta_k X)$  for  $k \in \{0, 1\}$  where the event type  $k = 0$  is censoring and event type  $k = 1$  is the event of type 1. The latent time for the event of second type is  $T_2$  and it is also drawn from an exponential distribution with parameters  $\lambda_2(t|X) = \lambda_2(t)\exp(\beta_2 \cos(X))$ . The observed event time is  $T = \min_{k \in \{0,1,2\}} \{T_k\}$  and the observed type of event is  $d = \arg \min_{k \in \{0,1,2\}} \{T_k\}$ . The parameters above are chosen as follows  $\lambda_1(t) = 1, \lambda_2(t) = 2, \beta_1 = 1, \beta_2 = 1$ . For  $\beta_0$ , we set two different values,  $\beta_0 = 0$  for the covariate independent censoring and  $\beta_0 = 1$  for the covariate dependent censoring. We set  $\lambda_0(t)$  such that the proportion of the right censored

data points is 50% and 75%. We compare the different models in terms of the existing metrics and the joint concordance at the 75% quantile of the times. We use three models for comparisons here: i) Cause-specific Cox model (CSC)<sup>3</sup> ii) Fine-Gray model (FG)<sup>4</sup> [13], and iii) the exponential model (EXP) ( $M(X, t, 1) = \exp(X)$ ,  $M(X, t, 2) = 2\exp(-\text{abs}(X))$ ), where  $\text{abs}(X)$  is the absolute value of  $X$ .

**Model comparisons** Our goal is to show that the existing metrics can lead to the selection of models that are bad for joint prediction of the event type and event time. First, we compute the exact values for all the metrics (concordance, accuracy, and joint concordance) using a large data set of 100,000 subjects for the synthetic experiment setup described above but in the absence of censoring. We compare the models in terms of standard metrics (concordance index for each event type [6] and the accuracy)  $\mathcal{V}(t)$ . We focus on the comparison between the CSC model and the EXP model. Based on the standard metrics (in Figure 2 and Table II), the CSC model seems to be better than the EXP model. However, when we compare the joint concordance, we find that the EXP model is better even though the CSC model Pareto dominates the EXP model in terms of existing metrics. EXP model has a 4% higher chance of correctly predicting both the event type and event time for a subject. We use the decomposition in (3) to get further insights into this comparison. The concordance conditional on accuracy for the EXP model is 0.74, and the accuracy is 0.70. The concordance conditional on accuracy for CSC model is 0.61, and the accuracy is 0.78. Although the CSC model can predict the event type in more cases, it is very poor in discriminating the subject for which it predicts the event correctly from other subjects. Poor discrimination implies that the event time predictions are also poor. Therefore, the CSC model is worse in comparison to the EXP model for the joint prediction of the event type and event time. Hence, the existing metrics can lead to poor model selection.

**Efficiency of the weighted estimators:** In this section, we evaluate the efficiency of the weighted estimators. We use the synthetic experiment setting described above that was also used to compare the models. We use the Kaplan Meier estimator for estimating the censoring distribution. We use the simulated datasets of size 1000 and 5000. We compute the RMSE, SE,

<sup>3</sup>We used the riskRegression package in R for the CSC model.

<sup>4</sup>We used the cmprsk package in R for the FG model.

TABLE I  
RMSE, SE AND BIAS FOR THE ESTIMATOR

Model	$\mathcal{JC}(t)$	Censoring	#samples	RMSE	SE	Bias
CSC	0.48	50 %	1000	0.0249	0.0247	0.0039
FG	0.46	50 %	1000	0.0338	0.0265	-0.0211
EXP	0.52	50 %	1000	0.0179	0.0160	0.0081
CSC	0.48	50 %	5000	0.0177	0.0137	0.0113
FG	0.46	50 %	5000	0.0259	0.0106	-0.0236
EXP	0.52	50 %	5000	0.0103	0.0067	0.0082
CSC	0.48	75 %	1000	0.0350	0.0338	-0.0095
FG	0.46	75 %	1000	0.0374	0.0319	-0.0192
EXP	0.52	75 %	1000	0.0308	0.0231	0.0205
CSC	0.48	75 %	5000	0.0231	0.0231	0.0020
FG	0.46	75 %	5000	0.0326	0.0189	-0.0266
EXP	0.52	75 %	5000	0.0202	0.0089	0.0180

TABLE II  
MODEL COMPARISONS IN TERMS OF EXISTING METRICS VS JOINT CONCORDANCE: SYNTHETIC DATA

Model	$\mathcal{C}(t, 1)$	$\mathcal{C}(t, 2)$	$\mathcal{A}(t)$	$\mathcal{JC}(t)$
FG	0.75	0.52	0.79	0.46
CSC	0.75	0.60	0.78	0.48
EXP	0.75	0.60	0.70	0.52

and the bias by averaging over 100 such datasets and compare them in Table I. All the comparisons that are carried out are in-sample. In Table I, we see that when the censoring rates are lower, the weighted estimator has a lower RMSE and bias.

### B. Real Data Experiments

In this section, we use real datasets to illustrate real use cases of the joint concordance index. We use cardiovascular risk dataset and SEER dataset.

**Cardiovascular Risks Dataset** The dataset [11] comprises of 1712 men (aged 40-59 years in 1960) from Italian Rural Areas of the Seven Countries Study [11]. During the 50-year follow-up, there were 12 different causes of death recorded: 318 due to Coronary Heart Disease (CHD), 162 to Heart Disease of Uncertain Etiology (HDUE), 225 to Stroke (STR) and another

TABLE III  
MODEL COMPARISONS IN TERMS OF EXISTING METRICS VS JOINT CONCORDANCE: CHD DEATHS VS HDUE DEATHS

Model	$\mathcal{C}(t, CHD)$	$\mathcal{C}(t, STR)$	$\mathcal{A}(t)$	$\mathcal{JC}(t)$
FG	0.76	0.79	0.56	0.45
CSC	0.76	0.79	0.55	0.43

TABLE IV  
MODEL COMPARISONS IN TERMS OF EXISTING METRICS VS PROPOSED METRIC: HDUE DEATHS VS STR DEATHS

Model	$\mathcal{C}(t, HDUE)$	$\mathcal{C}(t, STR)$	$\mathcal{A}(t)$	$\mathcal{JC}(t)$
FG	0.73	0.79	0.68	0.55
CSC	0.76	0.79	0.68	0.55

964 due to miscellaneous causes. Covariates measured at baseline were: Age, Arm Circumference, Cigarettes, Body Mass Index, Diabetes, Corneal Arcus, Serum Cholesterol, Blood Pressure, Heart Rate and Vital Capacity.

**SEER Datasets** We extracted 2 cohorts from the Surveillance, Epidemiology, and End Results (SEER) cancer registries, which cover approximately 28% of the US population [14]. In the first cohort the outcome is the time to cancer deaths versus the time to non-cancer deaths. In the second cohort, the outcome is time to digestive cancer deaths versus the time to breast cancer deaths. There are 81 covariates and here we specify some important of them: Age, Age at Diagnosis, Histology, Tumor size, Family history, Tumor grade, Race, etc.

**Model comparisons on cardiovascular risks dataset:** We estimate two models, the CSC model and the FG model. In the first comparison, the two competing events are the death due to CHD and the death due to STR. The comparisons in Table III reveal that the FG model and the CSC model are similar in terms of the standard concordance metrics and there is a small difference in terms of the accuracy. In this comparison, the FG model Pareto dominates the CSC model in terms of the existing metrics. We find that the FG model is also better in terms of the joint concordance.

For the second comparison given in Table IV, the two competing events are the death due to HDUE and the death due to STR. The comparisons in Table IV reveal that the CSC model Pareto dominates the FG model. However, there is no difference (statistically significant) between the joint concordance of both the models.

The takeaway from the comparisons in Tables III, IV is that a comparison in terms of the existing metrics is not sufficient to deduce the performance in terms of the joint concordance. Moreover, selecting a model based on the existing metrics can lead to selection of a model that performs worse in terms of joint prediction of event type and event time.

**Model comparisons on SEER datasets:** We estimate two models, the CSC model and the FG model. For the third comparison given in Table V, the two competing events are death due to cancer and death due to any other cause. The comparisons in Table V reveal that the CSC model and the FG model are similar in terms of the existing metrics. The CSC model performs better in terms of the joint concordance.

For the fourth comparison given in Table VI, the two competing events are death due to breast cancer and death due to digestive cancer. In terms of the existing metrics, the CSC model seems to perform better. In terms of the joint concordance, the FG model is better. From Table V, Table VI we can conclude that if one selected a model just based on the existing metrics then it can lead to selection of a poor model in terms of joint predictions.

**Variable Importance Ranking** In this section, our goal is to compare the standard approaches for variable importance ranking with the proposed stepwise CR regression (already described in the Section III-E). We used the same real dataset that we described in Section IV-B. We carry out two

TABLE V  
MODEL COMPARISONS IN TERMS OF EXISTING METRICS VS JOINT  
CONCORDANCE: CANCER DEATHS VS NON-CANCER DEATHS

Model	$\mathcal{C}(t, CAN)$	$\mathcal{C}(t, NCAN)$	$\mathcal{A}(t)$	$\mathcal{JC}(t)$
FG	0.71	0.61	0.70	0.47
CSC	0.71	0.61	0.70	0.50

TABLE VI  
MODEL COMPARISONS IN TERMS OF EXISTING METRICS VS JOINT  
CONCORDANCE: BREAST CANCER VS DIGESTIVE CANCER DEATHS

Model	$\mathcal{C}(t, BCAN)$	$\mathcal{C}(t, DCAN)$	$\mathcal{A}(t)$	$\mathcal{JC}(t)$
FG	0.82	0.87	0.72	0.82
CSC	0.83	0.87	0.93	0.78

comparisons: CHD deaths vs. STR deaths and HDUE deaths vs. STR deaths. We use the FG model to rank the risk factors.

In the first comparison given in Table VII, we compare the risk factor rankings when the two events are the CHD deaths and the STR deaths. We show that the ranking arrived at using the stepwise CR regression can be very different than the ranking arrived at using the standard approach based on the stepwise regression. We see that the proposed approach ranks cholesterol to be the highest, unlike the standard approach (cholesterol is ranked at seventh). Cholesterol is a strong event-specific risk-factor; it matters much more for the CHD deaths in comparison to the STR deaths (this is well known in the clinical literature [11][12]). The standard approach can miss such important risk factors. We also ranked the variables using the standardized regression based approach and we obtained the same conclusions.

In the second comparison in Table VIII, we compare the risk factor rankings when the two events are the HDUE deaths and the STR deaths. We show that the ranking arrived at using the joint concordance index is not very different in comparison to the ranking arrived at using the standard approach. This suggests that in this case for both the outcomes (HDUE deaths and STR deaths) the dataset does not contain risk factors that are only specific to one of the events.

Therefore, from Tables VII, VIII, we can see that in the cases when the dataset consists of risk factors that are exclusively specific to some events, the existing approaches can often fail to recognize their importance. On the other hand, the proposed approach is good at identifying the importance of these factors.

TABLE VII  
COMPARE VARIABLE IMPORTANCE: CHD VS. STR DEATHS

Rank	Stepwise regression	Stepwise CR regression
1	Blood Pressure	Cholesterol
2	Age	Blood pressure
3	Vital capacity	BMI
4	Arm circumference	Age
5	Corneal arcus	Diabetes

TABLE VIII  
COMPARE VARIABLE IMPORTANCE: HDUE VS. STR DEATHS

Rank	Stepwise regression	Stepwise CR regression
1	Age	Blood Pressure
2	Vital capacity	Age
3	Blood Pressure	Vital capacity
4	Arm circumference	Arm Circumference
5	Heart Rate	Heart Rate

## V. CONCLUSION

In SA-CR, existing metrics such as concordance and accuracy do not evaluate a model based on its joint prediction of the event type and event time. We have proposed a new metric that we call the joint concordance that overcomes the limitations of the existing metrics. We have proposed an estimator for the joint concordance that adjusts for the bias that occurs due to censoring and we prove that it is consistent. We have shown that the existing methods for variable importance ranking can often fail to recognize the importance of the event-specific risk factors, which are crucial for predicting the event type. We have introduced a new ranking method based on joint concordance that overcomes these limitations, which we call stepwise competing risks regression.

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## APPENDIX

### *Proof of Proposition 1*

Consider the case when there are two event types. We construct two models and compute their joint concordance and existing metrics.

The first model, i.e., Model 1, assigns the risk for a subject uniformly at random.  $R_i^1$  and  $R_i^2$  follow a uniform distribution over  $[0, 1]$ , where  $R_i^j$  is the risk for individual  $i$  for event type  $j$ . The risks for all the individuals are independent and identically distributed (i.i.d).

The second model, i.e., Model 2, assigns the risk  $R_i^1$  to subject  $i$  for event type 1 from a uniform distribution over  $[0, 1]$ . Model 2 assigns risk  $R_i^2 = 1 - R_i^1$  for the second event.

We compute concordance index of the Model 1 and the Model 2 for both event types as follows. We need to compute  $Pr(R_i^k > R_j^k | (T_i \leq t) \text{ and } (T_i < T_j \text{ or } D_j \neq k))$ .  $R_i^k - R_j^k$  is independent of  $(T_i \leq t) \text{ and } (T_i < T_j \text{ or } D_j \neq k)$ . Therefore,

$$Pr(R_i^k > R_j^k | (T_i \leq t) \text{ and } (T_i < T_j \text{ or } D_j \neq k)) = Pr(R_i^k > R_j^k) = \frac{1}{2} \quad (5)$$

A similar simplification also leads to the same values for the concordances for the Model 2.

We compute the accuracy for the Model 1. The accuracy for the event type 1 for Model 1 is given as follows

$$\begin{aligned} Pr(R_i^1 > R_i^2, D_i = 1 | T_i \leq t) &= \\ Pr(R_i^1 > R_i^2, D_i = 1) &= \frac{1}{2} Pr(D_i = 1 | T_i \leq t) \end{aligned} \quad (6)$$

Similarly, we compute the accuracy for event type 2 for Model 1 given as

$$\begin{aligned} Pr(R_i^2 > R_i^1, D_i = 2 | T_i \leq t) &= \\ Pr(R_i^2 > R_i^1, D_i = 2) &= \frac{1}{2} Pr(D_i = 2 | T_i \leq t) \end{aligned} \quad (7)$$

If we add the above two equations (6) and (7), we observe that the accuracy of the Model 1 is  $\frac{1}{2}$ .

We now compute the accuracy for Model 2. We use the definition of accuracy from the main manuscript to compute the accuracy for the event type 1 for Model 2 as follows

$$\begin{aligned} Pr(R_i^1 > 1 - R_i^1, D_i = 1 | T_i \leq t) &= \\ Pr(R_i^1 > 1 - R_i^1, D_i = 1) &= \frac{1}{2} Pr(D_i = 1 | T_i \leq t) \end{aligned} \quad (8)$$

Similarly, we can compute the accuracy for event type 2 for Model 2 and follow the same steps as that for Model 1 and find the accuracy to be equal to  $\frac{1}{2}$ .

We now compute the joint concordance for Model 2. Suppose Subject  $i$  experiences event type 1. Let us assume that the model correctly predicted the event type, i.e.,  $R_i^1 > \frac{1}{2}$ . Suppose Subject  $j$  experiences the event 1 at a later date or not at all. For Subject  $j$ , the value of the risk for event  $R_j^1$  is also a uniform random variable. We need to compute the probability  $Pr(R_j^1 < R_i^1 | R_i^1 > \frac{1}{2})$ . This probability can be written as the sum of two probabilities given as  $Pr(R_j^1 < \frac{1}{2} | R_i^1 > \frac{1}{2}) = \frac{1}{2}$  and  $Pr(R_j^1 > \frac{1}{2} \& R_i^1 < R_j^1 | R_i^1 > \frac{1}{2})$  given as

$$\begin{aligned} Pr(R_j^1 > \frac{1}{2} \& R_i^1 < R_j^1 | R_i^1 > \frac{1}{2}) &= \\ Pr(R_j^1 > \frac{1}{2}, R_j^1 > R_i^1 | R_i^1 > \frac{1}{2}) &= \frac{1}{4} \end{aligned} \quad (9)$$

Hence, the joint probability that the event type is predicted correctly and concordance condition also holds is

$$Pr(R_i^1 > \frac{1}{2}, R_j^1 < R_i^1) = \frac{1}{2}(\frac{1}{2} + \frac{1}{4}) = \frac{3}{8} \quad (10)$$

Therefore, the joint concordance for Model 2 is  $\frac{3}{8}$ . We now compute the joint concordance for Model 1.

$$\begin{aligned} Pr(R_i^1 > R_j^1 \& R_i^1 > R_i^2) &= \frac{1}{3} \\ Pr(R_i^2 > R_j^2 \& R_i^2 > R_i^1) &= \frac{1}{3} \end{aligned}$$

The above is true because  $R_i^1, R_i^2, R_j^1$  are i.i.d. These two conditions together with the fact that  $R_i^k$  are uniform random variables that are i.i.d. lead to the fact that joint concordance for Model 1 is  $1/3$ . Suppose that the claim in the Proposition is

false, i.e. there exists a function  $f$  such that  $\mathcal{JC}(t) = f(\mathcal{V}(t))$ . In the above examples, Model 1 and Model 2 have the same values for  $\mathcal{V}(t)$  equals to  $[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$ . If the above identity is true, then the joint concordance for Model 1 and Model 2 is  $f([\frac{1}{2}, \frac{1}{2}, \frac{1}{2}])$ . But based on the computations above, Model 1 and Model 2 have different values for the joint concordance (9). This leads to a contradiction. Therefore, there exists no such function. QED

**Proof of Proposition 2:** We want to simplify

$$Pr(D_i = d \& T_i \leq t \& (T_i < T_j \text{ or } D_i \neq D_j)) \quad (11)$$

We use  $I(s < T_j \text{ or } D_j \in d^c) = 1 - I(T_j \leq s \text{ and } D_j = d)$  in (11) to obtain

$$\begin{aligned} Pr(D_i = d \& T_i \leq t \& (T_i < T_j \text{ or } D_i \neq D_j) | X_i, X_j) \\ = \sum_d \int_0^t E_{X_i, X_j} \left[ (1 - F_d(s | X_j)) dF_d(s | X_i) \right] \end{aligned} \quad (12)$$

In the above (12),  $F_d$  is the CIF defined in Section III. We use this equation (12) to simplify the expression for joint concordance as shown in (13).

$\hat{G}$  is a consistent estimator for  $G$  (follows from the assumption that the censoring model is correctly specified). The weights  $\hat{W}_{ij}^1$  and  $\hat{W}_{ij}^2$  converge in probability to

$$W_{ij}^1 = \frac{1}{G(\tilde{T}_i - |X_i)G(\tilde{T}_i | X_j)} \quad (14)$$

$$W_{ij}^2 = \frac{1}{G(\tilde{T}_i - |X_i)G(\tilde{T}_i | X_j)} \quad (15)$$

From law of large numbers and Slutsky's lemma we can conclude that the estimator in (4) converges to (16). We simplify the inner terms in the numerator and denominator in the expression in (16) below.

$$\begin{aligned} E[A_{ij} W_{ij}^1 N_i(t, d) | X_i, X_j] \\ = \int_0^t G(s | X_j) S(s | X_j) G(s - |X_j) W_{ij}^1 dF_d(s | X_j) \\ = \int_0^t S(s | X_j) dF_d(s | X_j) \end{aligned} \quad (18)$$

$$\begin{aligned} E[B_{ij} W_{ij}^2 N_i(t, d) | X_i, X_j] \\ = \int_0^t \int_0^s G(s | X_j) S(s | X_j) G(s | X_j) W_{ij}^1 dF_d(s | X_j) \\ = \int_0^t F_{d^c}(s | X_j) dF_d(s | X_j) \end{aligned} \quad (19)$$

In (19),  $F_{d^c}$  is the CIF corresponding to the event other than  $d$ . Substituting the above equations (18) (19) in equation (16) we obtain (17). Since the expression in (17), (13) are the same we conclude that the weighted estimator is consistent.

$$\begin{aligned}
\mathcal{J}\mathcal{C}(t) &= Pr\left(M(X_i, t, D_i) > M(X_j, t, D_i) \& M_c(X_i, t) = D_i \mid T_i \leq t \& (T_i < T_j \text{ or } D_i \neq D_j)\right) \\
&= \sum_d \frac{E_{X_i, X_j} \left[ Pr\left(M(X_i, t, d) > M(X_j, t, d) \& M_c(X_i, t) = d \& D_i = d \& T_i \leq t \& (T_i < T_j \text{ or } D_i \neq D_j) \mid X_i, X_j\right) \right]}{Pr\left(T_i \leq t \& (T_i < T_j \text{ or } D_i \neq D_j)\right)} \\
\mathcal{J}\mathcal{C}(t) &= \sum_d \frac{E_{X_i, X_j} \left[ I\left(M(X_i, t, d) > M(X_j, t, d) \& M_c(X_i, t) = d\right) \int_0^t (1 - F_d(s|X_j)) dF_d(s|X_i) \right]}{\sum_d E_{X_i, X_j} \left[ \int_0^t (1 - F_d(s|X_j)) dF_d(s|X_i) \right]}
\end{aligned} \tag{13}$$

$$\begin{aligned}
\hat{\mathcal{J}}\mathcal{C}_{wtd}(t) &= \sum_d \frac{\sum_i \sum_j \left[ A_{ij} \hat{W}_{ij}^1 + B_{ij}(d) \hat{W}_{ij}^2 \right] N_i(t, d) I\left(M(X_i, t, d) > M(X_j, t, d) \& M_c(X_i, t) = d\right)}{\sum_d \sum_i \sum_j \left[ A_{ij} \hat{W}_{ij}^2 + B_{ij}(d) \hat{W}_{ij}^2 \right] N_i(t, d)} \\
&\rightarrow \sum_d \frac{E_{X_i, X_j} \left[ E \left[ \left[ A_{ij} W_{ij}^1 N_i(t, d) + B_{ij}(d) W_{ij}^2 N_i(t, d) \right] \mid X_i, X_j \right] I\left(M(X_i, t, d) > M(X_j, t, d) \& M_c(X_i, t) = d\right) \right]}{\sum_d E_{X_i, X_j} E \left[ \left[ A_{ij} W_{ij}^1 N_i(t, d) + B_{ij}(d) W_{ij}^2 N_i(t, d) \right] \mid X_i, X_j \right]}
\end{aligned} \tag{16}$$

$$\hat{\mathcal{J}}\mathcal{C}_{wtd}(t) \rightarrow \sum_d \frac{E_{X_i, X_j} \left[ \int_0^t (1 - F_d(s|X_j)) dF_d(s|X_i) I\left(M(X_i, t, d) > M(X_j, t, d) \& M_c(X_i, t) = d\right) \right]}{\sum_d E_{X_i, X_j} \left[ \int_0^t (1 - F_d(s|X_j)) dF_d(s|X_i) \right]} \tag{17}$$

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