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Working Alone and Working With Others: Implications for the Malthusian Era

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Abstract This paper presents a stylized dynamic model to study the impact of the social organization of production during the Malthusian Era (after the Neolithic Age and before the Industrial Revolution), during which there was little or no economic growth. The focus is on the division of time between working alone (individualism) and working with others (collectivism). This division of time matters because individuals have different productive abilities. A greater fraction of time spent working with others raises the income of current Low ability individuals – but it may also lower the income of High ability individuals and hence lower the bequests they leave for future Low ability individuals. In the presence of congestion effects, these forces interact in a very complicated way. The paper analyzes the comparative statics implications of this division of time on economic outcomes in the (unique, non-degenerate) Malthusian steady-state. It finds that a greater fraction of time spent working with others (a greater degree of collectivism) leads to a larger population, smaller per capita income and lower income inequality. Some historical evidence is consistent with these predictions.

Keywords division of time/labor · Malthusian epoch · steady state

JEL Classification: D00

1 Introduction

For thousands of years between the end of the Neolithic Age and the beginning of the Industrial Revolution –a period often called the *Malthusian Era* –there was virtually no economic growth: the mean growth rate of per capita income (GDP) for the world and for most societies was much less than 0.05%. (See (Clark 2008) for example.) But although there was no significant *growth* in per capita income *within* societies, there were significant – indeed substantial – differences in the *level* of per capita income *across* societies. These differences reflect the influence of many factors, including geography, technology and the social organization of production.

In this paper we focus on the last of these factors. We present a stylized, parsimonious dynamic model through which we explore the impact of the social organization of production – which we model as the division between time/labor spent working alone and time/labor spent working with others – on population size, GDP per capita and income inequality. In our model, the societal division of time/labor matters because individuals differ in productive ability. When working alone, output per unit time depends on individual ability; when working with others, output per unit time depends on the mean ability of society. This implies that the social division of time/labor produces two forces. On the one hand, time spent working with others (collectivism) provides a "social safety net" for *current* Low ability individuals. On the other hand, providing this "social safety net" uses some of the time/income of High ability individuals and hence decreases the bequests they leave to *future* Low ability individuals. Put differently: collectivism provides social insurance for individuals *at a given time*, but individualism provides social insurance for individuals *across time*. In the presence of congestion effects, the tension between these forces plays out in a complicated and subtle way. Formally, the dynamics of the system are governed by a coupled pair of partial differential equations with moving boundary conditions. We focus on and solve for the unique, non-degenerate steady state of this system. Our model suggests that technological differences across societies – and technological changes within societies – are important for population size but less important for per capita income or for income inequality, but that organizational differences across societies are important for all of these. These predictions are consistent with the literature and with some historical evidence.

As in the classic work of Solow (1956) we take the behavior of individuals as given in order to provide a cleaner analysis of the effects of that behavior on the outcomes of interest. We assume that all workers spend a fixed fraction of their time to working with others, consume a subsistence level of income and save the rest, which they leave as a bequest to subsequent generations. The division of time might arise from geographic, cultural or political factors. For a geographic example, consider the difference between ocean fishing and lake fishing. Ocean fishing requires greater collective effort than lake fishing, and so workers spend more time working with others in societies where ocean fishing is the predominant productive activity than in societies where lake fishing is the predominant productive activity (Liebbrant, Gneezy and List 2013). Similarly, hunting large animals (e.g., buffalo) requires greater collective effort than hunting small animals. For a cultural/political example, consider the findings of Greif's (1994) comparative study of Genovese and Maghribi traders, which showed that Maghribi traders operate their businesses in a very different way from Genovese traders: Maghribi traders actively solicit and share information and trading partners with other traders; Genovese traders do not. Greif's study also shows that Genovese traders earn higher income than Maghribi traders. Alternatively, it may be that *production* is not collectivist but *consumption* is. A richer, micro-founded model might explain the division of time – or the sharing of consumption – on the basis of optimizing behavior by individuals, but such a model is beyond the scope of this paper. Our model is silent about how differences across societies arise or how they persist – but the data does suggest that such differences did exist and did persist for long periods of time.

Clark (2007, 2008), Ashraf and Galor (2011) and Galor (2011) have offered mathematical models of the Malthusian period, but these models do not offer an explanation of how or why the social organization of production – or more generally social and cultural differences – might have influenced outcomes in this period. This is precisely the explanation our stylized model is intended to provide. Gorodnichenko and Roland (2011a, 2011b) offer an analysis of the impact of culture – especially the tension between individualism and collectivism – in the era *after* the Industrial Revolution. They argue that individualism rewards status and hence promotes innovation which in turn promotes growth. However it does not seem that this explanation can explain the impact of individualism vs. collectivism in the Malthusian Era – in which there was no growth. Benabou, Ticchi and Vindigni (2014, 2015) argue that a different aspect of culture – religiosity – has important implications for economic outcomes in *contemporary* societies but it is natural to speculate that religiosity might have had important implications in Malthusian societies as well. (Perhaps "tithing" might be thought of as a collective sharing of production.)

Following the theoretical developments, we compare our theoretical predictions with some historical evidence. To do this, we regress against Hofstede's (2011) measures of individualism, using Maddison's (2007) data on GDP and Peter and Williamson's (2007) data on inequality. We argue that our model, although certainly highly stylized, is at least consistent with this historical evidence: the R^2 for per capita income is 0.25 and the R^2 for the Gini coefficient (our measure of income distribution) is 0.63. We discuss our prediction about the irrelevance of technology in the context of the data and model offered by Ashraf and Galor (2011).

The remainder of the paper is organized in the following way. In Section 2 we begin with an expanded but still informal verbal description, and then lay out the formal mathematical model. Section 3 defines a steady state of the model, identifies assumptions on the parameters that are necessary and sufficient for the existence of a non-degenerate steady state, and discusses the meaning of those assumptions in the context of an illustrative example. (The detailed proof, which is relegated to the Appendix, solves for the steady state in closed form.) Section 4 collects the main theoretical predictions of our model. Section 5 discusses how the model might be extended to allow for ability to be heritable or inheritance to depend on ability. Section 6 presents and discusses some historical evidence. Section 7 concludes with some speculation about richer models. All proofs are collected in the Appendix.

2 Model

The features of the model that we develop here are intended to represent some aspects of societies in the Malthusian Era.

2.1 Informal Description

Before giving a formal mathematical description of the model, we begin with an informal verbal description that expands on what we have already said in the Introduction. We consider a world populated by a continuum of individuals of two types, either Low ability or High ability. Time is continuous and the horizon is infinite. The lifecycle of an individual is:

- the individual is born and comes into an inheritance;
- during its lifetime, the individual produces and consumes;
- the individual dies and leaves a bequest for succeeding individuals.

While they are alive and producing, each individual spends a fraction of its time working alone and consuming the output of its individual production, and the complementary fraction of its time working with others and sharing (equally) in the joint production. We view these fractions as the same across all individuals in the society, rather than as individual choices.[1] When individuals work alone, their output depends on their own ability; when individuals work with others, their output depends on the mean ability of society. In both modes, output is subject to congestion: productivity is less when the total population is greater. (The most obvious source of congestion is scarcity of natural resources; e.g. land. Congestion is an essential part of Clark's (2007, 2008) argument for why societies remain in the Malthusian trap; congestion plays an important role in our model as well.) During their lifetimes, individuals produce and consume. Some individuals produce less than they consume and eventually consume their entire inheritance; at that point their wealth is zero and they die in poverty. Individuals who do not die in poverty eventually die of natural causes. Individuals who die with positive wealth leave that wealth as a bequest to the new-born.

2.2 Formal Description

We now turn to the formal mathematical description. We consider a continuous time model with a continuum of individuals. Some individuals are of High ability and some are of Low ability; it is convenient to index by $Q = 0, 1$ (Low, High). The state of society at each moment of time is described by the *population distributions* $\mathcal{P}_0, \mathcal{P}_1$; $\mathcal{P}_Q(x,t)$ is the population of individuals of ability Q who have wealth less than or equal to x at time t. The *population of individuals of ability Q at time t* is

$$P_Q(t) = \lim_{x\to\infty} \mathcal{P}_Q(x,t)$$

Thus the *total population at time t* is

$$P(t) = P_0(t) + P_1(t)$$

[1] As suggested in the Introduction, we are agnostic about the origins of these fractions.

The productive ability of individuals of ability Q is π_Q.[2] The *mean productive ability of society at time t* is

$$\bar{\pi}(t) = \frac{P_0(t)\pi_0 + P_1(t)\pi_1}{P(t)}$$

Individuals are born at the constant (instantaneous) rate λ_b and die natural deaths at the constant (instantaneous) rate λ_d.[3] Half of all newborns are of High ability and half are of Low ability. (The assumption that half of new-borns are of High ability and half are of Low ability is made only for convenience. As we discuss in Section 5, the qualitative conclusions would remain the same if the fractions were different from one half, if ability were partly inheritable, and even if the birth and death rates those ofI High and Low ability were different.) As we discuss below, some individuals also die in poverty.

While they are alive, individuals produce and consume. We assume that each individual spends a fraction z of its time working alone and the remaining fraction $1-z$ working with others. When an individual works alone its production depends on its own ability and is consumed entirely by the individual; when it works with others its production depends on the mean ability of society (at the given moment of time) and is shared; in both modes, productivity is subject to congestion and so diminishes with increasing population. Production displays constant returns to scale (time spent working) so the production of an individual of ability $Q = 0, 1$ at a given time t when population is $P(t)$ and mean productive ability is $\bar{\pi}(t)$ is $[\pi_Q - cP(t)]$ when working alone and $[\gamma\bar{\pi}(t) - cP(t)]$ when working with others, where π_0, π_1 are the parameters of individual productivity, γ is the parameter of group productivity and $c > 0$ is a congestion parameter. Throughout, we assume that $c > 0$, reflecting the fact that some input – e.g., land – is in limited supply.[4] Hence the *overall productivity* of an individual of ability $Q = 0, 1$ is

$$\begin{aligned} F_Q(t) &= z[\pi_Q - cP(t)] + (1-z)[\gamma\bar{\pi}(t) - cP(t)] \\ &= z\pi_Q + (1-z)\gamma\bar{\pi}(t) - cP(t) \end{aligned} \tag{1}$$

We emphasize that Q is the innate and fixed ability of the (adult) individual and that $z, 1-z$ are characteristics of the society, and not individual choices.

For simplicity, we assume throughout that each individual consumes at a constant subsistence rate σ. We also assume that

$$\pi_0 < \sigma < \pi_1 \tag{2}$$

Thus, when working alone, Low ability individuals produce less than they consume and High ability individuals produce more than they consume. The rate of production net of consumption for an individual with ability Q is

$$H_Q(t) = F_Q(t) - \sigma = z\pi_Q + (1-z)\gamma\bar{\pi}(t) - [cP(t) + \sigma] \tag{3}$$

[2] We view individuals as productive adults so their productive ability is not changing over time.

[3] The assumption that individuals are born and die at constant rates is a simplification, following Blanchard (1985). Clark (2008) argues that the birth rate is an increasing function of the wealth of society and that the death rate is a decreasing function of the wealth of society. Those features could probably be incorporated into our model, although at the expense of mathematical complication. In the steady-state, which is our focus, birth and death rates would – by definition – be constant, but the birth and date rates would be determined endogenously.

[4] As we discuss later, if we allowed $c = 0$ then the steady state would be indeterminate.

Individuals who die at time t leave their wealth as a bequest to individuals born at the same time t. Of this bequest, a fraction $\eta \in [0,1]$ is passed on to newborns as their inheritance; the remaining fraction $1-\eta$ is lost in storage. (If $\eta = 1$ then storage is perfect and total bequests coincide with total inheritance.) We write $Y(t_0)$ as the (common) inheritance of individuals who are born at time t_0. So an individual of ability Q born at time t_0 begins life with *wealth* $X_Q(t_0) = Y(t_0)$; its wealth changes during its lifetime at the rate:

$$\frac{dX_Q(t)}{dt} = H_Q(t) \tag{4}$$

We stress that an individual's wealth may shrink or grow; if it shrinks, it may eventually shrink to 0 before the individual dies of natural causes in which case the individual dies in poverty. Of course individuals who die in poverty do not leave a bequest. In our analysis, we show that (under certain assumptions) the system has a unique non-degenerate steady state. As we shall show, in this steady state the wealth of Low ability individuals shrinks and the wealth of High ability individuals grows, and some Low ability individuals - but no High ability individuals - die in poverty.

We have defined the state of society at time t in terms of the population distributions $\mathcal{P}_0, \mathcal{P}_1$; however in analyzing the evolution of society it is more convenient to work with the *population densities* p_0, p_1. By definition,

$$\mathcal{P}_Q(x,t) = \int_0^x p_Q(\hat{x},t)\, d\hat{x}$$

Working with densities is more convenient because their evolution is determined by the following *Evolution Equations*:

$$\begin{aligned} \frac{\partial p_0(x,t)}{\partial x} H_0(t) + \frac{\partial p_0(x,t)}{\partial t} &= -\lambda_d\, p_0(x,t) \\ \frac{\partial p_1(x,t)}{\partial x} H_1(t) + \frac{\partial p_1(x,t)}{\partial t} &= -\lambda_d\, p_1(x,t) \end{aligned} \tag{EE}$$

To see how these partial differential equations arise, fix Q, x, t and consider the (infinitesimal) population of individuals of ability Q having wealth $x > 0$ at time t. By definition, the population density of such individuals is $p_Q(x,t)$. What changes for these individuals over an infinitesimal time interval Δt? Some of them die: because the instantaneous death rate is λ_d, the number who die is $\lambda_d p_Q(x,t)\Delta t$. For those who do not die, their wealth changes by $\Delta x = H_Q(t)\Delta t$. (Keep in mind that $H_Q(t)$ may be either positive or negative. If $H_Q(t) < 0$ then the wealth of these individuals is shrinking, but since Δt and hence Δx are infinitesimal, their wealth does not shrink to 0 during the time interval under consideration, and so none of these individuals die in poverty.) Hence the population density of individuals of ability Q having wealth $x + \Delta x$ at time $t + \Delta t$ is

$$p_Q(x+\Delta x, t+\Delta t) = p_Q(x,t) - \lambda_d p_Q(x,t)\Delta t \tag{5}$$

Because $\Delta t, \Delta x$ are infinitesimal, we can approximate:

$$\begin{aligned} p_Q(x+\Delta x, t+\Delta t) &\approx p_Q(x, t+\Delta t) + \left[\frac{\partial p_Q}{\partial x}(x, t+\Delta t)\right]\Delta x \\ p_Q(x, t+\Delta t) &\approx p_Q(x,t) + \left[\frac{\partial p_Q}{\partial x}(x,t)\right]\Delta t \end{aligned}$$

Putting these together, rearranging and assuming that p_Q is continuously differentiable yields

$$p_Q(x + \Delta x, t + \Delta t) - p_Q(x,t) \approx \left[\frac{\partial p_Q}{\partial x}(x,t)\right]\Delta x + \left[\frac{\partial p_Q}{\partial x}(x,t)\right]\Delta t$$

(5) tells us that $p_Q(x + \Delta x, t + \Delta t) - p_Q(x,t) = -\lambda_d p_Q(x,t)\Delta t$; recalling that $\Delta x = H_Q(t)\Delta t$ leads to

$$\begin{aligned} -\lambda_d p_Q(x,t)\Delta t &\approx \left[\frac{\partial p_Q}{\partial x}(x,t)\right]\Delta x + \left[\frac{\partial p_Q}{\partial x}(x,t)\right]\Delta t \\ &\approx \left[\frac{\partial p_Q}{\partial x}(x,t)\right] H_Q(t)\Delta t + \left[\frac{\partial p_Q}{\partial x}(x,t)\right]\Delta t \end{aligned}$$

Dividing through by Δt and taking limits gives the Evolution Equation (EE) for individuals of ability Q:

$$\frac{\partial p_Q(x,t)}{\partial x} H_Q(t) + \frac{\partial p_Q(x,t)}{\partial t} = -\lambda_d\, p_0(x,t)$$

Notice that neither deaths in poverty nor births appear in the Evolution Equations. This is because deaths in poverty only occur at $x = 0$ and births only occur at $x = Y(t)$ (inheritance at time t); the Evolution Equations are *not* presumed to hold at these values of x. Deaths in poverty and births enter into the behavior of the system as "boundary conditions" at $x = 0, x = Y(t)$. It is important to keep in mind that the "boundary" $x = Y(t)$ is *moving* because inheritance $Y(t)$ is a function of the population distributions and hence depends on time. It is also important to keep in mind that these Evolution Equations are *coupled* because net productivity $H_Q(t)$ of agents of ability Q depends on the *mean ability of the population* and hence on the *populations of each ability* and not just on the population of ability Q.

Unfortunately, such coupled systems of PDEs with moving boundary conditions are intractable; indeed, it is very difficult even to find good numerical approximations. Fortunately, the steady state of this system is governed by a system of coupled *ordinary differential equations*, and in the steady state the boundary $x = Y(t)$ is *not moving*, so we can solve in closed form.

3 Steady State

As discussed in the Introduction, we are most interested in societies in the steady state. We define a *steady state* as a state of the society in which the population density of individuals (of each type) across wealth levels is unchanging over time; i.e., $\partial p_Q/\partial t \equiv 0$ for $Q = 0, 1$. In a steady state, the birth and (overall) death rate are constant and equal, so the populations $P_0(t), P_1(t), P(t)$ are constant; write P_0^s, P_1^s, P^s for the steady state values. Because the population is constant, so are the mean ability $\pi^s = [P_0^s\pi_0 + P_1^s\pi_1]/P^s$, the net productivities of individuals of each ability $H_Q^s = zQ + (1-z)\pi^s - [cP^s + \sigma]$, and inheritance Y^s. (All these values will be determined endogenously by the parameters of the model and the condition that the society is in steady state.) Because the steady state distribution of individuals (of each type) across wealth levels is unchanging over time, we omit the time variable t and write the population density functions as functions of wealth level x

alone: $p_0(x), p_1(x)$. Note that because the dependence on time has disappeared, the Evolution Equations (EE) reduce to ordinary differential equations, so a steady state is a solution to the equations

$$\frac{dp_0(x)}{dx} H_0^s = -\lambda_d p_0(x) \tag{SSE0}$$

$$\frac{dp_1(x)}{dx} H_1^s = -\lambda_d p_1(x) \tag{SSE1}$$

that also satisfies the appropriate boundary conditions.[5] These boundary conditions are:

Boundary Conditions

B1 All individuals are born with the same inheritance Y^s.
B2 Half of all newborns are of High ability and half are of Low ability.
B3 The birth rate equals the total death rate.
B4 Total inheritance equals total bequests, discounted by η.

We focus on steady states in which the populations of both High and Low ability individuals are strictly positive and for which the productive output of both High and Low ability individuals are non-negative. (The latter restriction rules out steady states in which production is negative or resources are destroyed.) We refer to such steady states as *non-degenerate*. There is always a degenerate steady state in which population is identically 0. We show below that for certain parameter configurations there is also a degenerate steady state in which the population of Low ability individuals is 0. More importantly, for some parameter configurations no non-degenerate steady state exists, so we will need to impose various assumptions on parameters. Before doing so, it might be useful to illustrate the sorts of things that might go wrong.

– If $\lambda_b < \lambda_d$. then the population would shrink (and would shrink to 0 in the long run), so the society could not persist in a non-degenerate steady state.
– If $\lambda_b > 2\lambda_d$ then (because half of new-borns are of High ability and individuals of High quality never die in poverty) the population of High ability individuals would increase (and would blow up in the long run) so again society could not persist in a non-degenerate steady state.
– If $\lambda_b = 2\lambda_d$ then (because half of new-borns are of High ability and individuals of High quality never die in poverty) in order that the overall death rate be equal to the birth rate, it would necessarily be the case that *all* Low ability individuals die in poverty. This could only happen if inheritance $Y^s = 0$, in which case the population of Low ability individuals will be $P_0^s = 0$. (We have defined such a steady state to be degenerate.) This would be consistent with a positive population of High ability individuals only if $\eta = 0$ (so all bequests are lost entirely) or High ability individuals accumulate no wealth during their lifetimes; i.e. net productivity $H_1^s = z\pi_1 + (1-z)\gamma\pi_1 - [cP_1^s + \sigma] = 0$.
– If $z = 0$ then individuals work *only* with others. In such a society, individual output depends only on mean ability and not on individual ability, and hence net output in a steady state would be the same for all individuals; call this H^s.

[5] Keep in mind that these Steady State Equations remain *coupled*, because steady state net productivities H_0^s, H_1^s depend on the entire population and not just on the population of each ability separately.

If $H^s < 0$ then inheritance would be shrinking, which is incompatible with a steady state. If $H^s \geq 0$ then no individuals would be dying in poverty; if the natural birth rate strictly exceeds the natural death rate, this is incompatible with a non-degenerate steady state.[6]

– If $\eta = 0$ then individuals would be born with no inherited wealth. If Low ability individuals produced at least as much they consume then they would never die in poverty, so the the population could not be constant. If Low ability individuals produced less than they consume then they would die in poverty immediately so in a steady state their population would be zero. In either case there could not be a non-degenerate steady state.

As discussed above, for some configurations of the parameters there will be no non-degenerate steady state. However, in the presence of the assumptions that we maintain throughout ($\pi_0 < \sigma < \pi_1$ and $c > 0$), we can identify conditions that are necessary and sufficient for the existence of a non-degenerate steady state.[7] For convenience define

$$\beta = -\left(\frac{\lambda_d/\lambda_b}{\log\left[2 - (2\lambda_d/\lambda_b)\right]}\right)$$

$$\pi^* = \left[1 - \left(\frac{\lambda_b}{2\lambda_d}\right)\right]\pi_0 + \left[\frac{\lambda_b}{2\lambda_d}\right]\pi_1$$

(We will show in Theorem 2 that in fact $\pi^* = \pi^s$, the mean productivity of the population in the steady state.)

Assumptions

A1 $\lambda_d < \lambda_b < 2\lambda_d$
A2 $0 < z \leq 1$
A3 $\eta > 0$
A4

$$(1-z)\gamma\pi^* + z\left[\frac{\eta\beta}{1-\eta+\eta\beta}\right](\pi_1 - \pi_0) > \left(\frac{2\lambda_d}{\lambda_b}\right)(\sigma - \pi_0)$$

[6] It is worth noting that a society that cannot persist in a non-degenerate steady state might still persist in some oscillatory condition. For example, consider a purely collectivist society; i.e. $z = 0$. As noted above net output at time t is the same for all individuals, call it $H(t)$. Suppose that at some point t_0 in time, net output is strictly positive: $H(t_0) > 0$. At that moment t_0, no individuals are dying in poverty, so the population is growing and congestion is increasing, which implies that net output $H(t)$ is shrinking. Eventually $H(t)$ will become slightly negative, but because inheritance is positive, it will still be the case that few individuals die in poverty so population will still be growing. Eventually, congestion will reach the point at which individuals start dying in poverty often enough that population will begin to decline and $H(t)$ will begin to increase. Now there will be a race between population decline and production increase; the outcome will depend on the exact relationships among the parameters. One possibility is that population declines to the point that $H(t)$ becomes positive again, and the society oscillates back and forth between growth and decline; another possibility is that $H(t)$ remains negative and population eventually declines toward 0. Note that population cannot blow up because congestion would eventually overwhelm productivity entirely.

[7] If $c = 0$ then the steady state – if one exists at all – will be indeterminate: if p_0, p_1 are population densities that solve the steady state equations SSE0, SSE1 and satisfy the boundary conditions, then for every $K > 0$, Kp_0, Kp_1 will also be population densities that solve the steady state equations SSE0, SSE1 and satisfy the boundary conditions.

A5

$$z \leq \left(\frac{2\lambda_d}{\lambda_b}\right)\left(\frac{1-\eta+\eta\beta}{\eta\beta}\right) \Big/ (\pi_1 - \pi_0)$$

We have already discussed **A1 - A3**; we will discuss **A4** and **A5** in the context of an illustrative example in the next Subsection.

Theorem 1 *Assumptions* **A1 - A5** *are necessary and sufficient for the existence of a non-degenerate steady state. The non-degenerate steady state is unique.*

3.1 Illustrative Example

Our model is silent about the units in which the various parameters are measured; we offer here some choices that seem sensible. We think about the basic unit of time as one year and σ as the subsistence consumption for that year. Given that, it seems natural to measure individual productivities π_0, π_1 in comparison to σ. (The group productivity parameter is just a multiplier of the mean productivity of society.) Steady state production of individuals, per capita production of society and inheritance are also naturally measured in comparison to σ. Population might be measured in individuals or some multiple of individuals.

To illustrate, we computed the unique non-degenerate steady state for the following configuration of parameters:

$\sigma = 2$	$\pi_0 = 1$	$\pi_1 = 3$	$\eta = 1$	
$\lambda_b = 0.05$	$\lambda_d = 0.04$	$z = 1/2$	$\gamma = 1$	$c = 0.0001$

Note that, when working alone, High ability individuals produce 50% more than the subsistence level and Low ability individuals produce 50% less. (We have chosen $\sigma = 2$ rather than $\sigma = 1$ only so that π_0, π_1 will be whole numbers.) This configuration of parameters leads to a steady state in which the population is $P^s = 578$ (a large village). The population proportions are $P_0^s/P^s = 3/8$, $P_1^s/P^s = 5/8$ so 217 individuals are of Low ability and 361 are of High ability (numbers rounded off). Inheritance is $Y^s = 28.63$ (so that newborns come into enough inheritance to subsist for 14+ years without producing − but of course they do produce), $H_0^s = -0.625$ (so it actually takes 45.8 years for Low ability individuals to die in poverty) and $H_1^s = 0.375$. Keep in mind that H_0^s, H_1^s are production net of consumption $\sigma = 2$, so actual productions are $F_0^s = 1.375$ and $F_1^s = 2.375$. The per capita production of society (GDP per capita) is $F^s = 2.000$. (It is not a coincidence that GDP per capita F^s coincides with subsistence consumption σ. We will show in Theorem 5 that this is always the case when inheritance is perfect.)

Figure 1 illustrates the steady state wealth densities for this configuration of the parameters; the vertical line is inheritance, the left-hand curve is the population density of Low ability individuals, the right-hand curve is the population density of High ability individuals. Note that the wealth of all High ability individuals is above the inheritance level, that the wealth of all Low ability individuals is below the inheritance level, and that the total population of High ability individuals is greater than that of Low ability individuals. The population density of Low ability individuals is strictly positive at wealth $x = 0$ because some Low ability

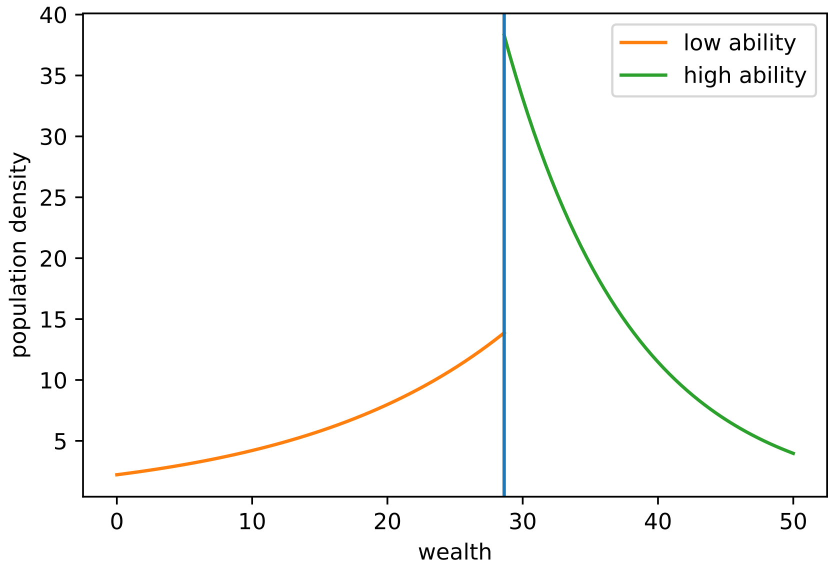

Fig. 1 Steady State Wealth Densities (Parameters Shown in the Text)

individuals die in poverty. The population density of High ability individuals is strictly positive for all wealths $x > Y^s = 28.63$, but only a portion of the curve is shown.

We have discussed the meaning of the Assumptions **A1 - A3**; this is a convenient place to say something about the meaning of **A4** and **A5**. Note that the left-hand-side of **A4** is a convex combination of $[\eta\beta/(1-\eta+\eta\beta)]\,(\pi_1 - \pi_0)$ and $\gamma\pi^*$, and the right-hand-side is a multiple of the amount by which Low ability individuals working alone fall short of the subsistence level. The inequality **A4** implies that at least one of $[\eta\beta/(1-\eta+\eta\beta)]\,(\pi_1 - \pi_0)$ and $\gamma\pi^*$ must be greater than $(2\lambda_d/\lambda_b)\,(\sigma - \pi_0)$; i.e., either group productivity is sufficiently large or the productive ability of High ability individuals is sufficiently greater than that of Low ability individuals or both. If only the first of these obtains, then $(1-z)$ cannot be too small; individuals must spend enough time working collectively. If only the second of these obtains, then z cannot be too small; individuals must spend enough time working individually. **A5** places an upper bound on the amount of time individuals can spend working individually; the extent to which this has bite depends on the productivity difference between High and Low ability individuals. We will see in the proof of Theorem 1 that **A4** is precisely what is required to guarantee that the population is strictly positive and **A5** is precisely what is required to guarantee that the production of Low ability individuals is non-negative.

Turning back to the Example again, let's think about the restrictions that **A4**, **A5** impose on z, keeping the other parameters fixed. Plugging in to **A4** we see that $\gamma\pi^* = 9/4$ and $[\eta\beta/(1-\eta+\eta\beta)]\,(\pi_1 - \pi_0) = 2$ so that both are bigger than $(2\lambda_d/\lambda_b)\,(\sigma - \pi_0) = 8/5$; hence **A4** imposes no restriction on z. However, plugging in to **A5** we find $z \leq 4/5$. In particular, for these values of the other parameters, a purely individualistic society could not exist in a non-degenerate steady state. (As we have already discussed, a purely collectivist society could never exist in a non-degenerate steady state.)

Finally, we note that the congestion parameter c does not enter into any of the Assumptions **A1-A5**. The reason is that the congestion parameter and the steady state population scale together, because the congestion effect that governs the steady state is the product cP^s. For example, suppose we halve the congestion parameter in the Example to $c_* = 500 = c/2$ but leave the other parameters unchanged. In the unique steady state of this new society, the population will double to $P^s_* = 1156 = 2P^s$ and the populations of High, Low ability individuals (and the population densities) will double as well, but all else will remain the same. In particular per capita inheritance will remain $Y^s = 28.63$ and production net of consumption will remain $H^s_0 = -0.625$, $H^s_1 = 0.375$.

4 Model Predictions

We now derive various implications for economic outcomes. In view of Theorem 1, there should be no confusion in referring to "the non-degenerate steady state."

We begin with the implications for the population fractions and then for the total population. The first two results may seem surprising; we will see that they are essentially accounting identities and are consequences of the assumption that natural deaths occur at a constant rate.

Theorem 2 *In the non-degenerate steady state the population fractions are*

$$\frac{P^s_0}{P^s} = 1 - \frac{\lambda_b}{2\lambda_d} \qquad\qquad \frac{P^s_1}{P^s} = \frac{\lambda_b}{2\lambda_d}$$

and the mean productive ability π^s of the population coincides with π^*, so:

$$\pi^s = \left(1 - \frac{\lambda_b}{2\lambda_d}\right)\pi_0 + \left(\frac{\lambda_b}{2\lambda_d}\right)\pi_1$$

Thus, the population fractions depend only on the birth and death rates λ_b, λ_d and are independent of the productivity parameters π_0, π_1, γ, the storage parameter η and the congestion parameter c; the mean productive ability of the population depends only only on the birth and death rates λ_b, λ_d and the productivity parameters π_0, π_1, and are independent of the group productivity parameter γ, the storage parameter η and the congestion parameter c.

Theorem 3 *In the non-degenerate state, the life expectancies of individuals depend only on the birth and death rates λ_b, λ_d and are independent of the productivity parameters π_0, π_1, γ, the inheritance parameter η and the congestion parameter c. Indeed, the survival functions of individuals depend only on λ_b, λ_d and are independent of π_0, π_1, γ, η and c.*[8]

Theorem 4 *In the non-degenerate steady state:*

(a) population is increasing in the productivity parameters π_0, π_1, γ and the bequest parameter η;

[8] The *survival function* for an individual specifies, for each time t, the probability that the individual is alive after time t.

(b) population is decreasing in the congestion parameter c;
– define

$$\gamma^* = \left[\frac{1}{\pi^s}\right]\left[\left(\frac{\eta\beta}{1-\eta+\eta\beta}\right)(\pi_1-\pi_0)+\left(\frac{2\lambda_d}{\lambda_b}\right)\pi_0\right]$$

(i) if $\gamma < \gamma^*$ then population is increasing in z;
(ii) if $\gamma > \gamma^*$ then population is decreasing in z.

A number of comments about these results are in order. The assertions of Theorem 4(a) all seem quite intuitive: a larger value for any of the technological parameters $\pi_0, \pi_1, \gamma, \eta$ – any improvement in technology – creates an *upward* force on the population. A larger value of π_0 means that Low ability individuals consume their endowment more slowly and hence tend to die in poverty less often. A larger value of π_1 means that High ability individuals accumulate greater wealth during their lifetimes and hence leave larger bequests, so Low ability individuals begin with a larger inheritance and hence tend to die in poverty less often. A larger value of γ increases the output of collective labor, so both Low and High ability individuals produce more; again, Low ability individuals tend to die in poverty less often. Finally, a larger value of η leads to larger bequests and hence to larger inheritance; again, Low ability individuals tend to die in poverty less often.

These intuitions are not wrong – but they are *incomplete*. To see why this is the case, it is useful to do a thought experiment and consider what would happen to a society that is in a steady state and received a positive shock to one of the technological parameters – e.g. π_0 (the productivity of Low ability individuals) – at some time t_0. Such a shock would disturb the steady state. Following the shock, *current* Low ability individuals (those who are alive at time t_0) would produce more and hence consume their current wealth more slowly than before and hence would be less likely to die in poverty; thus the population would increase and the proportion of Low ability individuals would also increase. But these would not be the only effects. As population increases, so does congestion. An increase in congestion reduces the productivity of all individuals, *including the High ability individuals.* This reduction in productivity means that the High ability individuals will leave smaller bequests when they die so that *future* Low ability individuals will begin life with a smaller inheritance. Evidently, these effects work in opposite directions: after the society settles into its new steady state, the productivity of Low ability individuals will be larger than in the original steady state, but their inheritance will be smaller. Theorem 4 tells us that the population will be larger than in the original steady state – but Theorems 2 and 3 tells us that the *population proportions* and the *life expectancies of individuals* will be the same.[9] Following a positive shock to π_1 (the productivity of High ability individuals), the adjustment process would be somewhat different but the end result would be similar. The shock would disturb the steady state. Following the shock, High ability individuals would produce more but consume the same amount and so leave larger bequests. Hence newborn Low ability individuals would begin life with a larger inheritance and be less likely to die in poverty. But this would increase the population and hence congestion, which would drive down the productive ability of

[9] As we have said, this is only a thought experiment, since we cannot prove that the society would actually settle into a new steady state. However, we believe it provides the correct intuition for the comparison of the two steady states.

Low ability individuals. Again, these effects work in opposite directions: after the society settles into its new steady state, the inheritance of Low ability individuals would be greater than in the original steady state, but their productivity would be less. Theorem 4(a) tells us that the population will be larger than in the original steady state – but Theorems 2 and 3 tell us that the population proportions and the lifespans of individuals will be the same.

Because this "invariance" of population proportions and of survival functions might seem so counter-intuitive, the reader might wonder if they are artifacts of our assumption that half of newborns are of High ability and half are of Low ability. As we discuss in Section 5, they are not: the same invariance would persist if the fractions were different, or if ability were heritable, and even if birth and death rates of High and Low ability individuals differed. As noted earlier, these results are essentially accounting identities.

As we have discussed in the context of the Illustrative Example in Subsection 3, the population scales inversely with the congestion parameter, which explains the assertion of Theorem 4(b).

The assertion of Theorem 4(c) also seems intuitive: if working with others is less productive for High ability individuals than working alone, then an increase in z, the amount of time spent working alone, increases the output of High ability individuals more than it decreases the output of Low ability workers so, as with an increase in π_1 as discussed above, this creates upward pressure on population. In the opposite direction, if working with others is more productive for High ability individuals than working alone then an increase in z decreases the output of both High and Low ability individuals and creates a downward pressure on population. Again, this intuition is not wrong but it is incomplete. An increase (decrease) in population leads to an increase (decrease) in congestion and hence to a decrease (increase) in the productivity of both High and Low quality individuals, which in turn creates a downward (upward) pressure on population. These effects do not balance out but they shift the threshold γ^* that separates the range in which population is increasing in z from the range in which population is decreasing in z. To see this shift, note that $\eta\beta/(1-\eta+\eta\beta) \leq 1$ and, by assumption, $\lambda_b < 2\lambda_d$, so $\gamma^*\pi^s > \pi_1$ and $\pi_1/\pi^s < \gamma^*$. For γ in the interval $(\pi_1/\pi^s, \gamma^*)$, population is increasing in z *even though High ability individuals produce more when working together than when working alone.* Because we have an explicit expression for this threshold γ^*, we can determine how this threshold changes with the parameters. For example, γ^* is increasing in the bequest factor η, which means that the interval $(\pi_1/\pi^s, \gamma^*)$ (where the population is increasing in z even though High ability individuals produce more when working together than when working alone) is getting bigger. Similarly, γ^* is increasing in the productivity π_0 of Low ability individuals and in the productivity π_1 of High ability individuals.

We now turn from population to income, in particular to mean income/GDP per capita and to income inequality. We identify income with productive output so, writing F_0^s, F_1^s for the steady state outputs of individuals of Low, High ability, respectively, the *mean capita income of society* or *GDP per capita* in the steady state is

$$F^s = [F_0^s P_0^s + F_1^s P_1^s]/P^s$$

Theorem 5 *In the non-degenerate steady state:*

(a) GDP per capita is independent of the congestion parameter c and the group productivity parameter γ;
(b) GDP per capita is decreasing in the bequest factor η;
(c) if $\eta < 1$ then GDP per capita is increasing in the fraction of time spent working alone z and the productivity difference $\pi_1 - \pi_0$;
(d) if $\eta = 1$ then GDP per capita is identically equal to subsistence σ and is independent of all the other parameters.[10]

The intuitions underlying (most of) these assertions are straightforward (although the actual analyses are less straightforward.) As we have commented in the Illustrative Example in Subsection 3, a change in the congestion factor simply scales the population but leaves the other fundamentals – in particular, individual production and hence GDP per capita – unchanged. That GDP per capita is independent of the group productivity parameter γ is a consequence of congestion: when group productivity goes up, so does population, and so does congestion, which reduces productivity; in the steady state the effects on production entirely cancel out. As we have noted in the Introduction, this is consistent with a general view of the Malthusian era that an improvement in technology (in our case, an increase in the group productivity factor γ) may lead in the short run to a temporary increase in GDP per capita but in the longer run simply to a larger population. (Ashraf and Galor (2011) provide data that supports this view.) Our model suggests a mechanism leading to this conclusion. (Ashraf and Galor (2011) offer a very different model that leads to the same conclusion.) In view of Theorem 4, an increase in the bequest factor η leads to an increase in the population and hence to greater congestion and lower GDP per capita, which is (b). That GDP is increasing in z and $\pi_1 - \pi_0$, as asserted by (c), seems intuitive. That the minimum – which necessarily occurs when $\eta = 1$ – is exactly σ (subsistence consumption), asserted in (d), seems striking.

Because a larger population means greater congestion, it might be tempting to think that an increase in the population would, of itself, lead to a decrease in individual production and mean production (GDP per capita), but the truth is more complicated. In our model, population cannot increase exogenously but only in response to a change in one or more parameters. If population increases because the Low ability individuals become more productive but all else remains the same, then the steady state production of Low ability individuals goes up but the steady state production of High ability individuals goes down, and the mean production (GDP per capita) goes down as well. If population increases because the High ability individuals become more productive but all else remains the same, the effect in the steady state is exactly the reverse: the production of Low ability individuals goes down, the production of High ability individuals goes up and mean production (GDP per capita) goes up as well. If time spent working alone z goes up but all else remains the same, population can move in either direction (depending on the regime of group productivity γ) but if inheritance is imperfect (i.e. $\eta < 1$) then mean production (GDP per capita) always goes up. All these can be seen from the expressions we derive for individual production and GDP per capita, equations (16), (17), (18) in the Appendix.

[10] Of course, High ability individuals produce more than σ and Low ability individuals produce less.

We measure income inequality in the familiar way as the Gini coefficient of the income distribution. Because there are only two types of individuals, the Gini coefficient is just the difference between the income share and the population share of the High ability individuals:

$$Gini = F_1^s P_1^s / F^s P^s - P_1^s / P^s = \left[\frac{P_1^s}{P^s} \right] \left[\frac{F_1^s}{F^s} - 1 \right]$$

Theorem 6 *In the non-degenerate steady state:*

(a) the Gini coefficient is increasing in the fraction of time spent working alone z and in the difference of individual productivities $\pi_1 - \pi_0$;
(b) the Gini coefficient is increasing in the bequest factor η;
(c) the Gini coefficient is independent of the group productivity parameter γ and the congestion parameter c;
(d) the Gini coefficient is less than $1 - (\lambda_b / 2\lambda_d) < 1/2$.

As with Theorem 5, the intuitions underlying (most of) these assertions are straightforward (although the actual analyses are less straightforward.) Note first that that the difference between the outputs of High and Low ability individuals is $F_1^s - F_0^s = z(\pi_1 - \pi_0)$. Hence an increase in either z or $\pi_1 - \pi_0$ results in an increase in the difference between the incomes of High/Low ability individuals and hence to an increase in the Gini coefficient, as asserted in (a). In view of Theorem 4, an increase in the bequest factor η leads to an increase in the population and hence to greater congestion and to lower overall income – but the difference in the incomes of the High/Low ability individuals remains the same, so the Gini coefficient increases. Because the population and hence the overall effect of congestion on production scales with c, the Gini coefficient is independent of c. Because GDP per capita is independent of γ and population proportions are fixed, the Gini coefficient is also independent of γ. Together, these assertions are (c). Finally, (d) is almost trivial: Theorem 2 tells us that the population share of High ability individuals is $\lambda_b / 2\lambda_d$, and the income share of High ability individuals is certainly less than 1, so the difference is certainly less than $1 - \lambda_b / 2\lambda_d$, which is (d).

5 Heritability

To this point we have assumed that half of all newborns are of High ability and half are of Low ability and that all individuals born at the same time obtain the same inheritance. Here we discuss the consequences of relaxing those assumptions.

5.1 Ability

It is natural to assume that ability is at least partly heritable; i.e. that the offspring of High ability individuals are more likely to be of High ability than are the offspring of Low ability individuals. To be specific, assume that the probability that the offspring of a High ability individual will be of High ability is $1 - \varepsilon_1$ and that the probability that the offspring of a Low ability individual will be of High ability is ε_0, with $0 < \varepsilon_0 < 1$ and $0 < \varepsilon_1 < 1$. It might also be natural to assume that birth rates depend on ability levels; say that the birth rate of Low ability individuals is λ_{b0} and the birth rate of High ability individuals is λ_{b1}. And we might also assume that natural death rates depend on ability levels; say that the natural death rate of Low ability individuals is λ_{d0} and the natural death rate of High ability individuals is λ_{d1}. We maintain the assumption that these birth and death rates are constant.

In this context, we obtain results analogous to those we have obtained before in Theorems 2 and 3.

Theorem 7 *In the context described above: in any non-degenerate steady state the population fractions are*

$$\frac{P_0^s}{P^s} = 1 - \left[\frac{\varepsilon_0\lambda_{b0}}{\lambda_{d1} + \varepsilon_0\lambda_{b0} - (1-\varepsilon_1)\lambda_{b1}}\right]$$
$$\frac{P_1^s}{P^s} = \frac{\varepsilon_0\lambda_{b0}}{\lambda_{d1} + \varepsilon_0\lambda_{b0} - (1-\varepsilon_1)\lambda_{b1}}$$

and the mean productive ability of the population is

$$\pi^s = \left(\frac{P_0^s}{P^s}\right)\pi_0 + \left(\frac{P_1^s}{P^s}\right)\pi_1$$

In particular, the population fractions depend only on the birth and death rates $\lambda_{b0}, \lambda_{b1}$, $\lambda_{d0}, \lambda_{d1}$ and the heritability rates $\varepsilon_0, \varepsilon_1$ and are independent of the productivity parameters π_0, π_1, γ, the inheritance parameter η and the congestion parameter c; the mean productive ability of the population depends only on the birth and death rates and the individual productivity parameters π_0, π_1, and is independent of the group productivity parameter γ, the inheritance parameter η and the congestion parameter c.

Theorem 8 *In the non-degenerate state, the life expectancies of individuals depend only on the birth and death rates $\lambda_{b0}, \lambda_{b1}, \lambda_{d0}, \lambda_{d1}$ and the heritability rates $\varepsilon_0, \varepsilon_1$, and are independent of the productivity parameters π_0, π_1, γ, the inheritance parameter η and the congestion parameter c. Indeed, the survival functions of individuals depend only on $\lambda_{b0}, \lambda_{b1}$, $\lambda_{d0}, \lambda_{d1}$ and are independent of π_0, π_1, γ, η and c.*

Because, even in this more general context, population proportions depend only on birth and death rates, we believe the same qualitative results (existence and uniqueness of the non-degenerate steady state and the economic implications) should obtain, with appropriate modifications to the assumptions **A4**, **A5** and the arguments.

5.2 Inheritance

It might also be natural to assume that newborns do not all obtain the same inheritance, but it is not entirely clear what form a more general assumption ought to take. One possibility is that newborns inherit only from their parents. This would imply that the offspring of Low ability individuals who have died in poverty die immediately; put differently: that Low ability individuals who die in poverty have no (surviving) offspring. Although this might seem strange, it does not seem to offer any particular difficulties in formulation or solution. A much bigger difficulty is that the offspring of individuals who die natural deaths will be born with different inheritances – indeed with a continuum of different inheritances. This would seem to make analysis intractable – or at least very hard.

Another possibility is that inheritance is tied to ability of the newborn (rather than to the parents). This seems strange, but perhaps not impossible: because we model the "newborns" as adults, rather than as infants, we might imagine that society would have had time to discover which individuals have which ability. Society could then decide to apportion the total available inheritance unequally according to revealed ability. If Low ability individuals were granted a larger inheritance – which might seem reasonable as a Rawlsian social norm – that would seem to create an incentive for High ability individuals to misrepresent themselves as Low ability, so if we were to insist on incentive compatibility, it would seem that society would have to grant a larger inheritance to High ability individuals. In a steady state we would then have two inheritance levels Y_0^s, Y_1^s with $Y_0^s < Y_1^s$ and we would have to determine *both* of these levels endogenously in the steady state, but this should not present any particular difficulties of analysis, and we believe the same qualitative implications would obtain, although the rule for apportioning inheritances according to ability (e.g., "apportion in a fixed ratio $Y_1^s/Y_0^s = \rho > 0$") would enter.

Yet a third possibility is that inheritance is tied to the ability of the parents, so that the offspring of High ability parents all receive the same inheritance and the offspring of Low ability parents all receive the same inheritance but that these inheritances differ according the the ability of the parents. In the steady state, this would seem to lead to a model in which there would be two kinds of individuals of each ability – those whose parents were Low ability and those whose parents were High ability – and hence solving the model would seem to mean solving four coupled differential equations, which does not seem an easy task.

6 Historical Evidence

As we have said before, we intend our model to be descriptive of societies in the Malthusian Era, the period between the Neolithic Revolution and the Industrial Revolution. Although only a limited amount of data is available for this period and there is some disagreement about its quality, it nevertheless seems appropriate to compare the predictions of our model with the data that is available.

Our model makes use of a number of parameters: the birth and death rates λ_b, λ_d, the fraction $1-\eta$ of bequests that are lost and not passed on to the newborn, the coefficient c of congestion, the group efficiency γ, and the fraction z of time spent working alone. Unfortunately, none of these parameters can be observed directly. (At least, none of these parameters were observed directly in the data that is available to us.) What *is* available is an index of individualism calculated by Hofstede (2011), which we use as a proxy for z (rescaled to lie in $[0,1]$).[11] In comparing the predictions of our model with historical data we ignore the cross-society differences between birth and death rates and the fraction of wealth that is heritable. It seems completely implausible to assume that technologies are the

[11] A natural alternative would be to assume that z is a (monotone) Box-Cox transformation (Box and Cox 1964) of Hofstede's index; in fact this would make almost no difference.

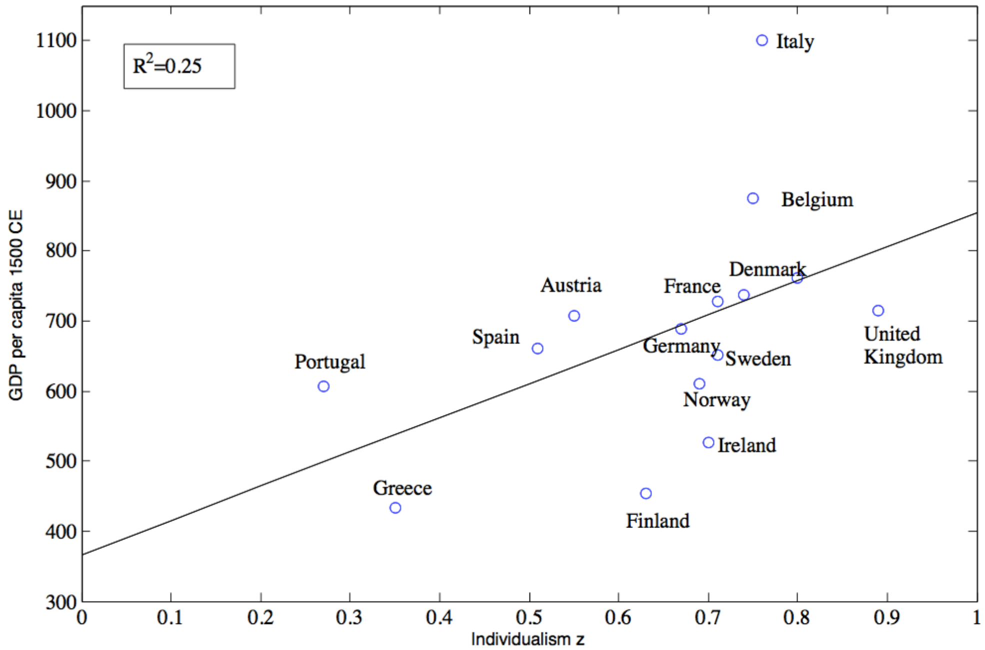

Fig. 2 GDP per capita vs. Individualism

same across societies – and hence that the technological parameters c, γ are the same across societies – so we focus on the predictions for GDP per capita and Gini coefficient, which are independent of these parameters.

To confront Theorem 5 with historical data, we use estimates of GDP in 1500 CE provided by Maddison (2007) for Western Europe. We use linear least-squares regression to compute the best-fitting straight line; see Figure 2. (Note that some of the "countries" that appear in Figure 2 – e.g. Italy – did not exist in 1500. Maddison uses the names to refer to the geographic areas occupied by the *current* countries.) Note that $R^2 = .25$ so that the theory explains a significant amount of the data.

Unfortunately, we do not find any data for Gini coefficients from 1500 CE (the period of the data used above). We therefore use the estimates of Gini coefficients given by Peter and Williamson (2007) from the (roughly) 100 year period 1788-1886 C.E., which might be thought to be after the Industrial Revolution. However, for those countries in which the Industrial Revolution arrived early (especially England, France and The Netherlands), the data and the calculations/estimations are from the beginning of this period, which would seem to be (mostly) *before* the (full impact of the) Industrial Revolution, while for those countries (especially Brazil, China and Peru) for which the data and the calculations/estimations are

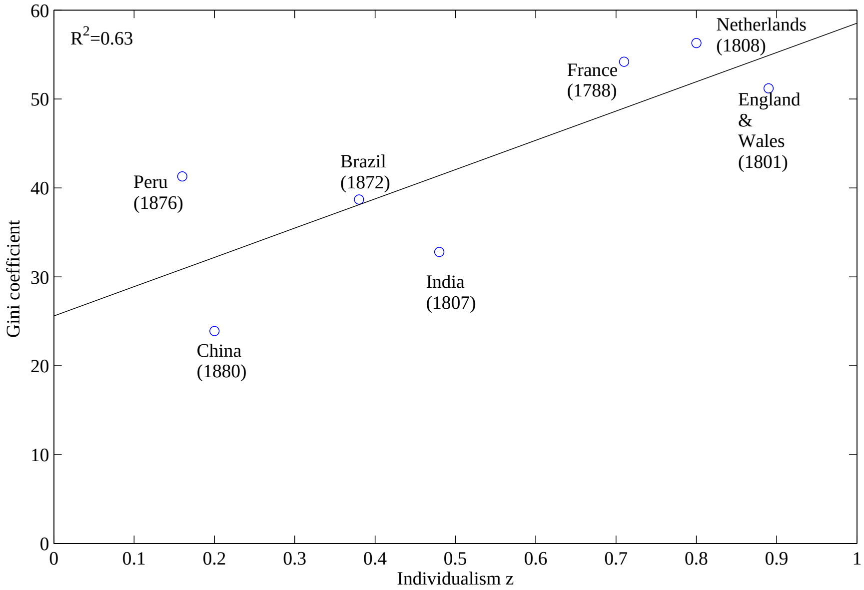

Fig. 3 Gini Coefficients vs. Individualism

from the end of this period, the Industrial Revolution did not in fact arrive until much later. The data and regression results can be seen in Figure 3. Note that $R^2 = .63$ so that the theory explains quite a bit of the data.

7 Discussion and Conclusions

This paper proposes and analyzes a model that provides a mechanism by which the tension between individualism and collectivism can lead to different economic outcomes in different societies. The model is intended to capture – albeit in a very stylized way – important features of the period between the Neolithic Revolution and the Industrial Revolution era as discussed in the work of Clark (2007, 2008) and others. The model makes predictions about the implications of the division of time/labor for economic outcome, and these predictions seem consistent with available historical data. In particular, we note that more time spent working alone leads to higher GDP per capita but also to greater inequality.

The model presented above makes many simplifying assumptions – but the model could be generalized in many dimensions. The most obvious generalization would allow for ability to be partly heritable and for birth and death rates to depend on ability. As we have shown, this would change the steady state population proportions, which would require corresponding changes in the Assumptions in order to guarantee that a non-degenerate steady state exists, but would not seem to lead to any qualitative changes in the implications. A more problematic generalization would allow for inheritance to depend on ability – but it is not quite clear what this should mean or how it should be modeled. If children inherit only from their parents, that would seem to lead to a setting in which the initial endowment of an individual would depend both on who their parents were and on when their parents died; such a setting would seem difficult to model and the resulting model would seem extremely difficult to solve. A more useful direction for generalization might be to allow for Low and High ability individuals to consume at different rates (this does not seem hard) or even at non-constant rates – changing through the lifecycle (this does seem hard). As Blanchard (1985) comments, the assumption that the natural rate of death is constant throughout an individual's lifetime does not not seem well-suited to modeling lifecycle effects.

We have confined our analysis to the non-degenerate steady state of the society which seems reasonable given that we are interested in the Malthusian Era in which there was little or no change. However even in the Malthusian era there were shocks – famines and epidemics – which perturbed the society from its steady state, so it would certainly be of interest to understand the stability properties of the model. From which perturbations would a society in the (non-degenerate) steady state return to the (non-degenerate) steady state? From which perturbations would the system collapse to the degenerate steady state? (As Diamond (2005) documents, such collapses are far from unknown.) Unfortunately, this an extremely complicated problem and well beyond our capabilities. Out of the steady state the dynamics of our model are governed by coupled PDE's with a moving boundary condition, and so the future evolution of the system depends on the entire wealth distribution and not just on a few aggregates. Such dynamics are well-known to be extremely resistant to rigorous analysis – or indeed, even to simulation, which are often extremely sensitive to small details of the numerical approximation.

As we have noted in at several places above, we do not model the division of time between working alone and working with others as arising from individual optimization. Despite this, we believe that the model presented here does offer some valuable insights. We leave the construction and analysis of a more micro-founded model – and the enormous challenges it will present – for future work.

Appendix

Because the proof of Theorem 1 is a bit roundabout we begin with a brief overview. By definition, a steady state is a pair of (non-zero) density functions $p_0(x), p_1(x)$ that satisfy the Steady State Equations (SSE0), (SSE1) and satisfy the Boundary Conditions **B1 - B4**. From the population densities $p_0^s(x), p_1^s(x)$ we can derive the following steady state quantities:

– the population of individuals with ability Q

$$P_Q^s = \int_0^\infty p_\pi^s(x)dx$$

– the total population

$$P^s = P_1^s + P_0^s = \int_0^\infty [p_1^s(x) + p_0^s(x)]dx$$

– mean productive ability

$$\pi^s = (\pi_0 P_0^s + \pi_1 P_1^s)/P^s$$

– net productivity of individuals of ability Q

$$H_Q^s = z\pi_Q + (1-z)\gamma\pi^s - cP^s - \sigma$$

– per capita wealth

$$X^s = \frac{\int_0^\infty x[p_1^s(x) + p_0^s(x)]dx}{\int_0^\infty [p_1^s(x) + p_0^s(x)]dx}$$

It is important to keep in mind that these quantities are constant and that they are determined entirely by the population densities p_0^s, p_1^s – whether or not these densities also satisfy the boundary conditions. Note that if the boundary condition **B4** holds then the steady state inheritance level Y^s determines per capita wealth X^s, so that the three steady state quantities P_0^s, P_1^s, Y^s determine all the others.

The population densities p_0^s, p_1^s determine the quantities P_0^s, P_1^s, Y^s; we proceed by showing that, subject to being strictly positive and satisfying a certain constraint, the quantities P_0^s, P_1^s, Y^s determine population densities p_0^s, p_1^s satisfying the Steady State Equations (SSE0), (SSE1). Once we have done that, we will be able to use the Boundary Conditions **B1 - B4** to pin down the values of the quantities P_0^s, P_1^s, Y^s that actually give rise to a steady state of the society. This will provide both the uniqueness and existence assertions of Theorem 1.

The constraint that we need arises from the following simple fact.

Lemma 1 *In every non-degenerate steady state, $H_0^s < 0 < H_1^s$; i.e., Low ability individuals produce less than they consume and High ability individuals produce more than they consume.*

Proof To prove this we show that the other possibilities are incompatible with a non-degenerate steady state. Note first of all that the definitions and the assumption that $0 < z \leq 1$ imply that Low ability individuals produce less than High ability individuals, so we must certainly have $H_0^s < H_1^s$. Hence there are only two possibilities that we need to rule out:

$\mathbf{0 \leq H_0^s < H_1^s}$ If this were the case then the wealth of Low ability individuals would be non-decreasing during their lifetimes and the wealth of High ability individuals would be strictly increasing during their lifetimes, so no individuals would die in poverty. Because the natural birth rate exceeds the natural death rate, this is impossible in a steady state..

$\mathbf{H_0^s < H_1^s \leq 0}$ If this were case then the wealth of Low ability individuals would be strictly decreasing during their lifetimes and the wealth of High ability individuals would be non-increasing during their lifetimes, so social wealth would be strictly decreasing, which is also impossible in a steady state.

We conclude that $H_0^s < 0 < H_1^s$ as asserted. □

As we have said, our claim about population proportions is essentially an accounting identity. Because we will make use of that accounting identity in the proof of Theorem 1, we jump to the proof of Theorem 2.

Proof of Theorem 2 By definition, the birth rate is λ_b and half of all newborns are of High ability, so in a small time interval Δt the mass of newly-born High ability individuals is approximately $[(1/2)\lambda_b P^s]\Delta t$. By definition, the death rate is λ_d. In the same small time interval Δt, the mass of High ability individuals who die a natural death is approximately $[\lambda_d P_1^s]\Delta t$. (No High ability individuals die in poverty because Lemma 1 guarantees that they produce more than they consume.) In the steady state, the population of High ability individuals is not changing, so these masses must be equal:

$$[\lambda_d P_1^s]\Delta t \approx [(1/2)\lambda_b P^s]\Delta t$$

Taking limits as $\Delta t \to 0$ and rearranging yields $P_1^s = (\lambda_b/2\lambda_d)P^s$. Since $P_0^s + P_1^s = P^s$, it follows that $P_0^s = 1 - (\lambda_b/2\lambda_d)P^s$, as asserted. □

We now return to the proof of Theorem 1. Our next task is to show how the quantities P_0^s, P_1^s, Y^s determine population densities p_0^s, p_1^s. To ease the notational burden a bit, we write

$$\lambda_0 = -\lambda_d/H_0^s \qquad\qquad \lambda_1 = \lambda_d/H_1^s$$

Lemma 2 *If P_0^s, P_1^s, Y^s are strictly positive and the derived productivities H_0^s, H_1^s satisfy the inequality $H_0^s < 0 < H_1^s$, then there are unique population densities p_0^s, p_1^s such that*

(a) p_0^s, p_1^s satisfy the Steady State Equations (SSE0), (SSE1) and the Boundary Condition **B1***;*
(b) the given populations P_0^s, P_1^s are derived from the population densities p_0^s, p_1^s; i.e.

$$P_0^s = \int_0^\infty p_0^s(x)dx$$
$$P_1^s = \int_0^\infty p_1^s(x)dx$$

The densities p_0^s, p_1^s are given by:

$$p_0^s(x) = \begin{cases} \left[\lambda_0 P_0^s/(1-e^{-\lambda_0 Y^s})\right] e^{\lambda_0(x-Y^s)} & \text{if } x < Y^s \\ 0 & \text{if } x > Y^s \end{cases} \tag{6}$$

$$p_1^s(x) = \begin{cases} \lambda_1 P_1^s\, e^{-\lambda_1(x-Y^s)} & \text{if } x > Y^s \\ 0 & \text{if } x < Y^s \end{cases} \tag{7}$$

(We leave p_0^s, p_0^s undefined at $x = Y^s$.)

Proof The condition $H_0^s < 0 < H_1^s$ means that the wealth of Low ability individuals is strictly decreasing while they are alive and the wealth of High ability individuals is strictly increasing while they are alive. Hence, solutions to the equations (SSE0), (SSE1) that correspond to the initial wealth level Y^s (i.e. satisfy **B1**) must satisfy $p_0^s(x) = 0$ for $x > Y^s$ and $p_1^s(x) = 0$ for $x < Y^s$; equivalently, p_0^s is supported on $[0, Y^s)$ and p_1^s is supported on (Y^s, ∞).

To determine p_0^s, note that for $x < Y^s$ the function p_0^s satisfies the ODE:

$$\frac{dp_0^s(x)}{dx} = \lambda_0 p_0^s(x)$$

The solution to this ODE is of the form

$$p_0^s(x) = C_0 e^{\lambda_0 (x - Y^s)}$$

The multiplicative constant C_0 is determined by the requirement that

$$\int_0^\infty p_0(x)dx = \int_0^{Y^s} p_0(x)dx = P_0^s$$

This forces $C_0 = \left[\lambda_0 P_0^s / (1 - e^{-\lambda_0 Y^s})\right]$ so we have (6).

To determine p_1^s, note that for $x > Y^s$, the function p_1^s satisfies the ODE:

$$\frac{dp_1^s(x)}{dx} = -\lambda_1 p_1^s(x)$$

The solution to this ODE is of the form

$$p_1^s(x) = C_1 e^{-\lambda_1 (x - Y^s)}$$

The multiplicative constant C_1 is determined by the requirement that

$$\int_{Y^s}^\infty p_1(x)dx = P_1^s$$

This forces $C_1 = \lambda_1 P_1^s$ so we have (7). □

We now derive a second implication of the boundary condition **B2**: half of all newborns are of High ability and half are of Low ability.

Lemma 3 *In every non-degenerate steady state*

$$\lim_{x \downarrow Y^s} p_1^s(x) H_1^s = (\lambda_b/2) P^s = -\lim_{x \uparrow Y^s} p_0^s(x) H_0^s$$
$$e^{-\lambda_0 Y^s} = 2 - \frac{P^s}{P_1^s}$$

(As usual, $\lim_{x \downarrow Y^s}$ *and* $\lim_{x \uparrow Y^s}$ *are the limits from above, below.)*

Proof To see that $p_1^s(x)H_1^s = (\lambda_b/2)P^s$, fix an arbitrary initial time t and small amount of time Δt and consider the population of High ability individuals who were born during the time interval $[t, t+\Delta t]$. Because the instantaneous birth rate is λ_b and half of all newborns are of High ability, the mass of this population is $(\lambda_b/2)P^s\Delta t$. Of this population, some die; because High ability individuals never die in poverty and the instantaneous rate of natural deaths is λ_d, the mass of those who die is approximately $[(\lambda_b/2)P^s\Delta t]\Delta t$. Those who do not die were born with inheritance Y^s and accumulated wealth at the rate H_1 so they comprise the population of individuals whose wealth lies between Y^s and $Y^s + H_1\Delta t$. If Δt is small, this mass is approximately $p(Y^s + H_1\Delta t)(H_1\Delta t)$. We have computed the mass of the same population in two different ways so we obtain

$$(\lambda_b/2)P^s\Delta t - [(\lambda_b/2)P^s\Delta t]\Delta t \approx p(Y^s + H_1\Delta t)(H_1\Delta t)$$

Dividing by Δt and taking the limit as $\Delta t \to 0$ yields the desired equality for High ability individuals.

To obtain the corresponding equality for Low ability individuals, consider the population of Low ability individuals who were born during the time interval $[t, t+\Delta t]$. Observe that $Y^s > 0$ so that if Δt is small then no Low ability individuals die in poverty during this time interval. The remainder of the argument parallels the argument for High ability individuals in the obvious way, keeping in mind that $H_0 < 0$ so the the wealth of Low ability individuals is decreasing.

To obtain the last equality, substitute the solutions obtained in Lemma 2; this yields

$$\lambda_1 P_1^s H_1^s = -\frac{\lambda_0 P_0^s H_0^s}{1 - e^{-\lambda_0 Y^s}}$$

Recall that $\lambda_1 = \lambda_d/H_1^s$ and $\lambda_0 = -\lambda_d/H_0^s$, so after various cancellations we are left with

$$P_1^s = \frac{P_0^s}{1 - e^{-\lambda_0 Y^s}}$$

Re-arranging and remembering that $P_0^s + P_1^s = P^s$ gives the last equality. □

Our next task is to derive the implications of **B3**: in a steady state, the birth rate and overall death rate must be equal

Lemma 4 *In every non-degenerate steady state, the instantaneous rate μ^s at which Low ability individuals die in poverty satisfies*

$$\lambda_b P^s = \mu^s P_0^s + \lambda_d P^s$$
$$\mu^s = -\lim_{x\downarrow 0} p_0(x)H_0^s$$

Proof The first equality is simply the boundary condition that, in a steady state, the birth rate must equal the total death rate, together with the observation that only Low ability individuals die in poverty. To obtain the second equality, we proceed as in Lemma 3 by fixing an arbitrary time t and considering a small time interval Δt. The mass of Low ability individuals who die in poverty in between 0 and $t + \Delta t$ is approximately $\mu^s P_0^s\Delta t$. As before, we can calculate this mass in another way by noting that, because the wealth of Low ability individuals decreases at the rate H_0^s, the individuals who die in poverty during this period are precisely those whose wealth at time t lies in the interval $[0, H_0^s \Delta t]$ and who do not die of natural causes during that time. Keeping in mind that $H_0^s < 0$, the total mass of Low ability individuals whose wealth is in that interval is approximately $[-p_0^s(H_0^s \Delta t) H_0^s]$ so the mass of those who die of natural causes is approximately $[-p_s^0(H_0^s)\Delta t][H_0^s \Delta t]\lambda_d \Delta t$; the remainder die in poverty. Hence we have

$$\mu^s P_0^s \Delta t \approx -p_0(H_0^s \Delta t) H_0^s \Delta t - [-p_0(H_0^s \Delta t) H_0^s \Delta t]\lambda_d \Delta t$$

Dividing by Δt and taking limits as $\Delta t \to 0$ yields the second equality, so the proof is complete. □

With these results in hand we can give the proof of Theorem 1.

Proof of Theorem 1 In the steady state, the boundary condition **B4** is that the inheritance of the new-born equals the bequests left by the dying:

$$Y^s = \eta \left(\frac{\lambda_d}{\lambda_b} \right) X^s \tag{8}$$

By definition, the steady state per-capita wealth X^s of the population is

$$X^s = \frac{1}{P^s} \left\{ \int_0^{Y^s} x p_0(x) dx + \int_{Y^s}^{\infty} x p_1(x) dx \right\}$$

Substituting for the solutions p_0^s, p_1^s found in Lemma 2, using the fact that $e^{-\lambda_0 Y^s} = 2 - P^s/P_1^s$ and simplifying yields

$$X^s = \frac{P_1^s}{P^s} \left[\int_0^{Y^s} \lambda_0 x e^{\lambda_0 (x - Y^s)} dx + \int_{Y^s}^{\infty} x \lambda_1 e^{-\lambda_1 (x - Y^s)} dx \right]$$

Integration by parts yields:

$$\int_0^{Y^s} \lambda_0 x e^{\lambda_0 (x - Y^s)} dx = \frac{e^{-\lambda_0 Y} - 1}{\lambda_0}$$
$$\int_{Y^s}^{\infty} \lambda_1 x e^{-\lambda_1 (x - Y^s)} dx = \frac{1 + \lambda_1 Y^s}{\lambda_1}$$

Substituting and expanding yields

$$X^s = \left(\frac{\lambda_b}{\lambda_d} \right) Y + \left(\frac{1}{\lambda_d} \right) [z\pi_0 + (1 - z)\gamma \pi^s] + \left(\frac{1}{\lambda_d} \right) \left(\frac{\lambda_b}{2\lambda_d} \right) [z(\pi_1 - \pi_0)] - \left(\frac{1}{\lambda_d} \right) (cP^s + \sigma)$$

In view of the boundary condition (8) we can write $X^s = (1/\eta)(\lambda_b/\lambda_d)Y^s$; substituting and collecting terms yields

$$\left(\frac{1-\eta}{\eta}\right)\left(\frac{\lambda_b}{\lambda_d}\right)Y = \left(\frac{1}{\lambda_d}\right)[z\pi_0 + (1-z)\gamma\pi^s] + \left(\frac{1}{\lambda_d}\right)\left(\frac{\lambda_b}{2\lambda_d}\right)[z(\pi_1 - \pi_0)] - \left(\frac{1}{\lambda_d}\right)(cP^s + \sigma)$$

Solving for $cP^s + \sigma$ yields

$$cP^s + \sigma = \left(\frac{\eta - 1}{\eta}\right)(\lambda_b Y) + [z\pi_0 + (1-z)\gamma\pi^s] + \left(\frac{\lambda_b}{2\lambda_d}\right)[z(\pi_1 - \pi_0)] \tag{9}$$

Lemma 3 tells us that $e^{-\lambda_0 Y^s} = (2 - P^s/P_1^s)$; taking logarithms, substituting $\lambda_0 = -\lambda_d/H_0$ and multiplying by $-H_0$ yields

$$\lambda_d Y^s = \left[\log\left(2 - \frac{2\lambda_d}{\lambda_b}\right)\right][z\pi_0 + (1-z)\gamma\pi^s - (cP^s + \sigma)] \tag{10}$$

Using (9) to substitute for $cP^s + \sigma$ and observing that various terms cancel, we obtain

$$\lambda_d Y^s = \left[\ln\left(2 - \frac{2\lambda_d}{\lambda_b}\right)\right]\left[\left(\frac{\lambda_b}{2\lambda_d}\right)z(\pi_1 - \pi_0) + \left(\frac{1-\eta}{\eta}\right)\lambda_b Y^s\right]$$

We can now solve for Y^s. In order to arrive at a more compact expression, set

$$\beta = -\left[\frac{\lambda_d/\lambda_b}{\ln\left(2 - \frac{2\lambda_d}{\lambda_b}\right)}\right]$$

Then we have the desired expression for Y^s:

$$Y^s = \frac{\eta\beta(\pi_1 - \pi_0)}{2\lambda_d(1 - \eta + \eta\beta)} \tag{11}$$

If we plug into (9) and solve for P^s we obtain:

$$P^s = \frac{1}{c}\left\{\frac{\lambda_b}{2\lambda_d}\left(\gamma\pi^s + z\left[\frac{\eta\beta(\pi_1 - \pi_0)}{1 - \eta + \eta\beta} - \gamma\pi^s + \left(\frac{2\lambda_d}{\lambda_b}\right)\pi_0\right]\right) - \sigma\right\} \tag{12}$$

Recall that Theorem 2 tells us that

$$\pi^s = \frac{P_0^s\pi_0 + P_1^s\pi_1}{P^s} = \left(1 - \frac{\lambda_b}{2\lambda_d}\right)\pi_0 + \left(\frac{\lambda_b}{2\lambda_d}\right)\pi_1$$

so we have an expression for P^s that involves only the given parameters of the society. We have already solved for Y^s in (11) and shown in Theorem 2 that

$$P_0^s = \left(1 - \frac{\lambda_b}{2\lambda_d}\right)P^s \qquad\qquad P_1^s = \left(\frac{\lambda_b}{2\lambda_d}\right)P^s \tag{13}$$

We have therefore shown that for any configuration of parameters, the requirement that the society be in a non-degenerate steady state completely determines the populations P_0^s, P_1^s and the inheritance level Y^s, and therefore the population densities p_0^s, p_1^s. Hence for any configuration of strictly positive parameters P_0^s, P_1^s, Y^s for which the constraint $H_0^s < 0 < H_1^s$ is satisfied, there is at most one non-degenerate steady state.

However we are not quite done because we have not identified those configurations of parameters that actually lead to a *non-degenerate* steady state. To do this, we have to show that if our Assumptions **A1-A5** hold then the expressions we obtain for the populations P_0^s, P_1^s and the inheritance Y^s are actually strictly positive, that the expressions we obtain for the net levels of production H_0^s, H_1^s satisfy the inequality $H_0^s < 0 < H_1^s$, and that the expressions we obtain for the levels of production F_0^s, F_1^s are also strictly positive.

We begin with Y^s; see (11). By assumption, $\pi_1 > \pi_0$, $0 < z \leq 1$, $0 < \eta \leq 1$. By construction, $\beta > 0$. Hence Y^s is strictly positive for all configurations of parameters.

Because P_0^s, P_1^s are non-zero fractions of P^s, we need only check that P^s is strictly positive. This will be the case exactly when

$$\left(\gamma\pi^s + z\left[\frac{\eta\beta(\pi_1 - \pi_0)}{1-\eta+\eta\beta} - \gamma\pi^s + \left(\frac{2\lambda_d}{\lambda_b}\right)\pi_0\right]\right) > \left(\frac{2\lambda_d}{\lambda_b}\right)\sigma$$

If we rewrite this as

$$(1-z)\gamma\pi^s + \left[\frac{\eta\beta}{1-\eta+\eta\beta}\right] z(\pi_1 - \pi_0) > \left(\frac{2\lambda_d}{\lambda_b}\right)(\sigma - \pi_0)$$

and recall that we have already shown in Theorem 2 that $\pi^* = \pi^s$, we see that the desired inequality is exactly **A4**.

Next, we show that $H_0^s < 0 < H_1^s$. To obtain the first inequality, recall that $H_0^s = z\pi_0 + (1-z)\gamma\pi^s - (cP^s + \sigma)$ and note that this is precisely the second term in square brackets in equation (10). Hence

$$\begin{aligned} H_0^s &= \frac{\lambda_d Y^s}{\left[\log\left(2 - \frac{2\lambda_d}{\lambda_b}\right)\right]} \\ &= -\left(\frac{\lambda_b}{2\lambda_d}\right)\left(\frac{\eta\beta}{1-\eta+\eta\beta}\right)[z(\pi_1 - \pi_0)] \end{aligned} \tag{14}$$

This is certainly negative: $H_0^s < 0$. Note that $H_1^s = H_0^s + z(\pi_1 - \pi_0)$. Plugging in to (14) yields

$$H_1^s = -\left(\frac{\lambda_b}{2\lambda_d}\right)\left(\frac{\eta\beta}{1-\eta+\eta\beta}\right)[z(\pi_1 - \pi_0)] + z(\pi_1 - \pi_0) \tag{15}$$

By assumption, $\lambda_b/2\lambda_d < 1$, and $\eta\beta/(1-\eta+\eta\beta)$ is certainly no greater than 1, so $H_1^s > 0$, as desired. Thus $H_0^s < 0 < H_1^s$ for all configurations of parameters.

Finally, we must show that F_0^s, F_1^s are non-negative. Since $F_0^s = H_0^s + \sigma$, we have

$$F_0^s = -\left(\frac{\lambda_b}{2\lambda_d}\right)\left(\frac{\eta\beta}{1-\eta+\eta\beta}\right)[z(\pi_1 - \pi_0)] + \sigma$$

After re-writing to isolate z, we see that $F_0^s \geq 0$ exactly when

$$z \leq \left(\frac{2\lambda_d}{\lambda_b}\right)\left(\frac{1-\eta+\eta\beta}{\eta\beta}\right) \Big/ (\pi_1 - \pi_0)$$

This is exactly **A5**, so $F_0^s \geq 0$. Because $F_1^s > F_0^s$, F_1^s is certainly positive, so the proof is complete. □

Proof of Theorem 3 In a non-degenerate steady-state, High ability individuals die only of natural causes and never die in poverty. Since natural death is a Poisson arrival process with intensity λ_d the probability that a High ability individual will have died by time $T > 0$ is just $\int_0^T (1/\lambda_d)e^{-\lambda_d t}dt$ so the probability that a High ability individual will be alive after time $T > 0$ is

$$\mathbb{P}_1(T) = 1 - \int_0^T (1/\lambda_d)e^{-\lambda_d t}dt$$

Low ability individuals die in poverty provided they live long enough. The natural death rate is λ_d and the death rate in poverty is μ^s so the probability of a Low ability individual dying in poverty is $\mu^s/(\mu^s + \lambda_d)$. Because their inheritance is Y^s and their wealth is changing at the rate $dX_0/dt = H_0^s$, the wealth of Low ability individuals will shrink to 0 at time $T^* = -Y^s/H_0^s$, at which point they will die in poverty. Before that time, Low ability individuals may die of natural causes. Because the natural death process for Low ability individuals is the same as for High ability individuals, the probability that a Low ability individual dies before time $T \leq T^*$ is just $\int_0^T (1/\lambda_d)e^{-\lambda_d t}dt$. If a Low ability individual has not died by time T^* then it dies in poverty at that time. Hence we must have

$$\left[\int_0^{T^*} (1/\lambda_d)e^{-\lambda_d t}dt\right] + \left[\frac{\mu^s}{\mu^s + \lambda_d}\right] = 1$$

To solve for T^* first integrate

$$\int_0^{T^*} (1/\lambda_d)e^{-\lambda_d t}dt = -e^{-\lambda_d T^*} + 1$$

We know from Lemma 4 that $\mu^s = (\lambda_b - \lambda_d)(P^s/P_0^s)$ and from Lemma 3 that $P^s/P_0^s = 1/(1 - \lambda_b/2\lambda_d)$; plugging in and simplifying yields

$$\begin{aligned}\left[\frac{\mu^s}{\mu^s + \lambda_d}\right] &= \frac{(\lambda_b - \lambda_d)/(1 - \lambda_b/2\lambda_d)}{(\lambda_b - \lambda_d)/[(1 - \lambda_b/2\lambda_d) + \lambda_d]} \\ &= 2 - \frac{2\lambda_d}{\lambda_b}\end{aligned}$$

Hence

$$\begin{aligned}e^{-\lambda_d T^*} &= \left(2 - \frac{2\lambda_d}{\lambda_b}\right) \\ T^* &= -\frac{1}{\lambda_d}\log\left(2 - \frac{2\lambda_d}{\lambda_b}\right)\end{aligned}$$

(Recall that, by assumption, $\lambda_d < \lambda_b < 2\lambda_d$ so $0 < (2\lambda_d/\lambda_b) - 1 < 1$ and $\log[(2\lambda_d/\lambda_b) - 1] < 0$, so $T^* > 0$.) Hence we conclude that the probability $\mathbb{P}_0(T)$ that a Low ability individual will be alive after time T is

$$\mathbb{P}_0(T) = \begin{cases} 1 - \int_0^T (1/\lambda_d) e^{-\lambda_d t} dt & \text{if } T < -\frac{1}{\lambda_d} \log\left(2 - \frac{2\lambda_d}{\lambda_b}\right) \\ 0 & \text{if } T \geq -\frac{1}{\lambda_d} \log\left(2 - \frac{2\lambda_d}{\lambda_b}\right) \end{cases}$$

We have just shown that the survival functions for both High and Low quality individuals depend only on the death and birth rates, and are independent of the other parameters. It follows that the life expectancies of both High and Low quality individuals depend only on the death and birth rates, and are independent of the other parameters, so the proof is complete. □

Proof of Theorem 4 In the proof of Theorem 1 above, (12) provides an explicit expression for P^s:

$$P^s = \frac{1}{c}\left\{\frac{\lambda_b}{2\lambda_d}\left(\gamma\pi^s + z\left[\frac{\eta\beta(\pi_1 - \pi_0)}{1 - \eta + \eta\beta} - \pi^s\gamma + \left(\frac{2\lambda_d}{\lambda_b}\right)\pi_0\right]\right) - \sigma\right\}$$

where $\beta = -(\lambda_d/\lambda_b)/\ln(2 - \frac{2\lambda_d}{\lambda_b})$. It is immediate that P^s is decreasing in c and increasing in π_1, γ. The coefficient of π_0 is

$$z\left[1 - \left(\frac{\lambda_b}{2\lambda_d}\right)\left(\frac{\eta\beta}{1 - \eta + \eta\beta}\right)\right]$$

By assumption, $\lambda_b/2\lambda_d$ is less than 1, and $\eta\beta/((1 - \eta + \eta\beta)$ is certainly no greater than 1, so the coefficient of π_0 is strictly positive and so P^s is also increasing in π_0. Because $z > 0$, calculation shows that the derivative of P^s with respect to η is positive, so P^s is increasing in η. Finally, it is evident that P^s is increasing in z if

$$\pi^s\gamma < \frac{\eta\beta(\pi_1 - \pi_0)}{1 - \eta + \eta\beta} + \left(\frac{2\lambda_d}{\lambda_b}\right)\pi_0$$

and decreasing in z if the inequality is reversed, so

$$\gamma^* = \left(\frac{1}{\pi^s}\right)\left[\left(\frac{\eta\beta(\pi_1 - \pi_0)}{1 - \eta + \eta\beta}\right) + \left(\frac{2\lambda_d}{\lambda_b}\right)\pi_0\right]$$

is the desired threshold, as asserted. This completes the proof. □

Proof of Theorem 5 We identify GDP per capita with mean income F^s. Equations (14), (15) give us explicit formulas for H_0^s, H_1^s so we obtain explicit formulas for F_0^s, F_1^s:

$$F_0^s = -\left(\frac{\lambda_b}{2\lambda_d}\right)\left(\frac{\eta\beta}{1 - \eta + \eta\beta}\right)[z(\pi_1 - \pi_0)] + \sigma \quad (16)$$

$$F_1^s = -\left(\frac{\lambda_b}{2\lambda_d}\right)\left(\frac{\eta\beta}{1 - \eta + \eta\beta}\right)[z(\pi_1 - \pi_0)] + z(\pi_1 - \pi_0) + \sigma \quad (17)$$

By definition, GDP per capita is $F^s = (P_0^s/P^s)F_0^s + (P_1^s/P^s)F_1^s$. We know that $P_1^s/P^s = \lambda_b/2\lambda_d$ and $P_0^s/P^s = 1 - \lambda_b/2\lambda_d$; after doing the requisite algebra we find that GDP per capita is:

$$F^s = \left[\frac{\lambda_b}{2\lambda_d}\right]\left[1 - \frac{\eta\beta}{1-\eta+\eta\beta}\right] z(\pi_1 - \pi_0) + \sigma \tag{18}$$

There is another way to arrive at the same result which may shed some additional light. In a steady state, aggregate wealth is unchanging, so what is produced must be equal to what is consumed plus whatever is lost in transforming bequests to inheritance. Total (instantaneous) production is P^sF^s; total (instantaneous) consumption is $P^s\sigma$; total (instantaneous) bequests are $P^s(\lambda_d X^s)$ and loss is $(1-\eta)P^s(\lambda_d X^s)$. The boundary condition **B4** tells us that $\lambda_b Y^s = \eta(\lambda_d X^s)$ so loss is $(1-\eta)P^s(\lambda_b/\lambda_d)(1/\eta)Y^s$. Hence

$$P^sF^s = P^s\sigma + (1-\eta)P^s(\lambda_b/\lambda_d)(1/\eta)Y^s$$

Dividing by P^s, substituting the expression for Y^s from (11) and doing the requisite algebra again yields (18).

It is evident that that GDP per capita F^s is independent of the group productivity parameter γ and the congestion parameter c, and decreasing in the bequest factor η. It is (separately) linear in the degree of individualism z and the productivity difference $(\pi_1 - \pi_0)$, so if $\eta < 1$ then GDP per capita is increasing in the degree of individualism z and the productivity difference $\pi_1 - \pi_0$; if $\eta = 1$, GDP per capita is σ and is independent of all the other parameters. □

Proof of Theorem 6 Some preliminary manipulation will help to clarify the dependence of the Gini coefficient on the various parameters.

$$\begin{aligned}
Gini &= \left[\frac{P_1^s}{P^s}\right]\left[\left(\frac{F_1^s}{F^s}\right) - 1\right] \\
&= \left[\frac{P_1^s}{P^s}\right]\left[\frac{F_1^s - F^s}{F^s}\right] \\
&= \left[\frac{P_1^s}{P^s}\right]\left[\frac{F_1^s - \left[(P_0^s/P^s)F_0^s + (P_1^s/P^s)F_1^s\right]}{F^s}\right] \\
&= \left[\frac{P_1^s}{P^s}\right]\left[\frac{F_1^s - F_0^s}{F^s}\right]\left[1 - \left(\frac{P_1^s}{P^s}\right)\right] \\
&= \left[\frac{\lambda_b}{2\lambda_d}\right]\left[1 - \left(\frac{\lambda_b}{2\lambda_d}\right)\right]\left[\frac{z(\pi_1 - \pi_0)}{(\lambda_b/2\lambda_d)\left[1 - \frac{\eta\beta}{1-\eta+\eta\beta}\right] z(\pi_1 - \pi_0) + \sigma}\right]
\end{aligned}$$

It is evident that the Gini coefficient is independent of group productivity γ and congestion c and increasing in individualism z, the bequest factor η and the difference $\pi_1 - \pi_0$, as asserted.

Finally, recall that the Gini coefficient is the difference between the income share and the population share of the High ability individuals. Since their population share is $\lambda_b/2\lambda_d$ and their income share is less than 1, the Gini coefficient is less than $1 - \lambda_b/2\lambda_d$, as asserted. □

Proof of Theorem 7 Fix a steady state in which the populations of Low, High ability individuals is P_0^s, P_1^s. The proportion ε_0 of the offspring of Low ability individuals will be of High ability and the proportion $1-\varepsilon_1$ of the offspring of High ability individuals will be of High ability; in any small time interval Δt the mass of newborn High ability individuals will be approximately $[\varepsilon_0 \lambda_{b0} P_0^s + (1-\varepsilon_1)\lambda_{b1} P_1^s]\Delta t$. The mass of High ability individuals who die in the same time interval will be approximately $[\lambda_{d1} P_1^s]\Delta t$. Because we are in steady state these must be equal:

$$[\varepsilon_0 \lambda_{b0} P_0^s + (1-\varepsilon_1)\lambda_{b1} P_1^s]\Delta t \approx [\lambda_{d1} P_1^s]\Delta t$$

Keeping in mind that $P_0^s + P_1^s = P^s$, taking limits as $\Delta t \to 0$ and performing the requisite algebra, we obtain

$$\frac{P_1^s}{P^s} = \frac{\varepsilon_0 \lambda_{b0}}{\lambda_{d1} + \varepsilon_0 \lambda_{b0} - (1-\varepsilon_1)\lambda_{b1}}$$

Of course $P_0^s/P^s = 1 - P_1^s/P^s$. The assertion about mean productive ability is just the definition.[12] □

Proof of Theorem 8 The argument is exactly the same as in the proof of Theorem 3. The survival function of the High ability individuals depends only on their natural death rate; the survival function of the Low ability individuals depends only on their natural birth and death rate and the time it takes for them to die in poverty, which is completely determined by their proportion in the steady state. □

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[12] Note that the expression for P_1^s reduces (of course) to $P_1^s = (\lambda_b/2\lambda_d)P^s$ when $\varepsilon_0 = \varepsilon_1 = 1/2$, $\lambda_{b0} = \lambda_{b1} = \lambda_b$ and $\lambda_{d0} = \lambda_{d1} = \lambda_d$, as in the previous text.

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